

Stochastic Gravity: A Primer with Applications

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Abstract

Stochastic semiclassical gravity of the 90's is a theory naturally evolved from semiclassical gravity of the 70's and 80's. It improves on the semiclassical Einstein equation with source given by the expectation value of the stress-energy tensor of quantum matter fields in curved spacetimes by incorporating an additional source due to their fluctuations. In stochastic semiclassical gravity the main object of interest is the noise kernel, the vacuum expectation value of the (operator-valued) stress-energy bi-tensor, and the centerpiece is the (stochastic) Einstein-Langevin equation. We describe this new theory via two approaches: the axiomatic and the functional. The axiomatic approach is useful to see the structure of the theory from the framework of semiclassical gravity, showing the link from the mean value of the energy momentum tensor to their correlation functions. The functional approach uses the Feynman-Vernon influence functional and the Schwinger-Keldysh close-time-path effective action methods which are convenient for computations. It also brings out the open systems concepts and the statistical and stochastic contents of the theory such as dissipation, fluctuations, noise and decoherence. We then describe the applications of stochastic gravity to the backreaction problems in cosmology and black hole physics. In the first problem we study the backreaction of conformally coupled quantum fields in a weakly inhomogeneous cosmology. In the second problem we study the backreaction of a thermal field in the gravitational background of a quasi-static black hole (enclosed in a box) and its fluctuations. These examples serve to illustrate closely the ideas and techniques presented in the first part. This article is intended as a first introduction providing readers with some basic ideas and working knowledge. Thus we place more emphasis here on pedagogy than completeness. [Further discussions of ideas, issues and on-going research topics can be found in [1, 2, 3] respectively.]

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1 From Semiclassical to Stochastic Gravity

The first step in the road to stochastic gravity begins with *quantum field theory in curved spacetimes* (QFTCST) [4, 5, 6, 7, 8], which describes the behavior of quantum matter fields propagating in a specified (not dynamically determined by the quantum matter field as source) background gravitational field. In this theory the gravitational field is given by the classical spacetime metric determined from classical sources by the classical Einstein equations, and the quantum fields propagate as test fields in such a spacetime. For time dependent spacetime geometry it may not be possible to define a physically meaningful vacuum state for the quantum field at all times. Assuming there is an initial time the vacuum state at a latter time will differ from that defined initially because particles are created in the intervening time. An important process described by QFTCST is indeed particle creation from the vacuum (and effects of vacuum fluctuations and polarizations) in the early universe [9] and Hawking radiation in black holes [10, 11].

The second step in the description of the interaction of gravity with quantum fields is backreaction, *i.e.*, the effect of the quantum fields on the spacetime geometry. The source here is the expectation value of the stress-energy operator for the matter fields in some quantum state in the spacetime, a classical observable. However, since this object is quadratic in the field operators, which are only well defined as distributions on the spacetime, it involves ill defined quantities. It contains ultraviolet divergences the removal of which requires a renormalization procedure. The ultraviolet divergences are already present in Minkowski spacetime, but in a curved background the renormalization procedure is more involved since it needs to preserve general covariance [4, 12]. The final expectation value of the stress-energy operator using a reasonable regularization technique is essentially unique, modulo some terms which depend on the spacetime curvature and which are independent of the quantum state. This uniqueness was proved by Wald [13, 14] who investigated the criteria that a physically meaningful expectation value of the stress-energy tensor ought to satisfy.

The theory obtained from a self-consistent solution of the geometry of the spacetime and the quantum field is known as *semiclassical gravity*. Incorporating the backreaction of the quantum matter field on the spacetime is thus the central task in semiclassical gravity. One assumes a general class of spacetime where the quantum fields live in and act on, and seek a solution which satisfies simultaneously the Einstein equation for the spacetime and the field equations for the quantum fields. The Einstein equation which has the expectation value of the stress-energy operator of the quantum matter field as the source is known as the *semiclassical Einstein equation* (SEE). Semiclassical gravity was first investigated in cosmological backreaction problems [15, 16], an example is the damping of anisotropy in Bianchi universes by the backreaction of vacuum particle creation. Using the effect of quantum field processes such as particle creation to explain why the universe is so isotropic at the present was investigated in the context of chaotic cosmology [17] in the late seventies prior to the inflationary cosmology proposal of the eighties [18], which assumes the vacuum expectation value of a Higg's field as the source, another, perhaps more well-known, example of semiclassical gravity.

1.1 The importance of quantum fluctuations

For a free quantum field semiclassical gravity is robust in the sense that it is consistent and fairly well understood. The theory is in some sense unique, note that the only reasonable c-number stress-energy tensor that one may construct [13, 14] with the stress-energy operator is a renormalized expectation value. However the scope and limitations of the theory are not so well understood. It is expected that the semiclassical theory would break down at the Planck scale. One can conceivably assume that it would also break down when the fluctuations of the stress-energy operator are large [19, 20]. Calculations of the fluctuations of the energy density for Minkowski, Casimir and hot flat spaces as well as Einstein and de Sitter universes are available [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. It is less clear, however, how to quantify what a large fluctuation is, and different criteria have been proposed [20, 33, 34, 22, 23, 35]. The issue of the validity of the semiclassical gravity viewed in the light of quantum fluctuations is summarized in our Erice lectures [2]. One can see the essence of the problem by the following example inspired by Ford [19].

Let us assume a quantum state formed by an isolated system which consists of a superposition with equal amplitude of one configuration of mass M with the center of mass at X_1 and another configuration of the same mass with the center of mass at X_2 . The semiclassical theory as described by the semiclassical Einstein equation predicts that the center of mass of the gravitational field of the system is centered at $(X_1 + X_2)/2$. However, one would expect that if we send a succession of test particles to probe the gravitational field of the above system half of the time they would react to a gravitational field of mass M centered at X_1 and half of the time to the field centered at X_2 . The two predictions are clearly different, note that the fluctuation in the position of the center of masses is of the order of $(X_1 - X_2)^2$. Although this example raises the issue of how to place the importance of fluctuations to the mean, a word of caution should be added to the effect that it should not be taken too literally. In fact, if the previous masses are macroscopic the quantum system decoheres very quickly [36] and instead of being described by a pure quantum state it is described by a density matrix which diagonalizes in a certain pointer basis. For observables associated to such a pointer basis the density matrix description is equivalent to that provided by a statistical ensemble. The results will differ, in any case, from the semiclassical prediction.

In other words, one would expect that a stochastic source that describes the quantum fluctuations should enter into the semiclassical equations. A significant step in this direction was made in Ref. [37] where it was proposed to view the back-reaction problem in the framework of an open quantum system: the quantum fields seen as the “environment” and the gravitational field as the “system”. Following this proposal a systematic study of the connection between semiclassical gravity and open quantum systems resulted in the development of a new conceptual and technical framework where (semiclassical) Einstein-Langevin equations were derived [38, 39, 40, 41, 42]¹ The key technical factor to most of these results was the use of the influence functional method of Feynman and Vernon [43] when only the coarse-grained effect of the environment on the system is of interest.

However, although Einstein-Langevin equations were derived for several models, the results were somewhat formal and some concern could be raised on the physical reality of the solutions of the stochastic equations for the gravitational field. This is related to the issue of the environment

¹The word semiclassical put in parentheses here refers to the fact that the noise source in the Langevin equation arises from the quantum field, while the background spacetime is classical. We will not carry this word in connection with the ELE or stochastic gravity, since there is no confusion that the source which contributes to the stochastic features of this theory comes from quantum fields.

induced quantum to classical transition. In the language of the consistent histories formulation of quantum mechanics [44] for the existence of a semiclassical regime for the dynamics of the system one needs two requirements: The first is decoherence, which guarantees that probabilities can be consistently assigned to histories describing the evolution of the system, and the second is that these probabilities should peak near histories which correspond to solutions of classical equations of motion. The effect of the environment is crucial, on the one hand, to provide decoherence and, on the other hand, to produce both dissipation and noise to the system through back-reaction, thus inducing a semiclassical stochastic dynamics on the system. As shown by different authors [45, 46] (indeed over a long history predating the current revival of decoherence) stochastic semiclassical equations are obtained in an open quantum system after a coarse graining of the environmental degrees of freedom and a further coarse graining in the system variables. It is expected but has not yet been shown that this mechanism could also work for decoherence and classicalization of the metric field, since a quantum description of the gravitational field is lacking. Thus far, the analogy could be made formally [47] or under certain assumptions (such as adopting the Born-Oppenheimer approximation in quantum cosmology [48]).

Later an axiomatic approach to the Einstein-Langevin equation without invoking the open system analogy was suggested based on the formulation of self-consistent dynamical equation for a perturbative extension to semiclassical gravity able to account for the lowest order stress-energy fluctuations of matter fields [49]. It was then shown that the same equation could be derived, in this general case, from the influence functional of Feynman and Vernon [27]. The field equation is deduced via an effective action which is computed assuming that the gravitational field is a c-number. It is interesting to note that the Einstein-Langevin equation can also be understood as a useful intermediary tool to compute symmetrized two-point correlations of the quantum metric perturbations on the semiclassical background, independent of a suitable classicalization mechanism [50]. The important new element in the derivation of the Einstein-Langevin equation, and of the stochastic gravity theory, is the physical observable that measures the stress-energy fluctuations, namely, the expectation value of the symmetrized bi-tensor constructed with the stress-energy tensor operator: the *noise kernel*.

1.2 An illustrative model

Before embarking on the formulation of stochastic gravity let us illustrate the theory with a simple toy model which minimalizes the technical complications. The model will be useful to clarify the role of the noise kernel and illustrate the relationship between the semiclassical, stochastic and quantum descriptions. Let us assume that the gravitational equations are described by a linear field $h(x)$ whose source is a massless scalar field $\phi(x)$ which satisfies the Klein-Gordon equation in flat spacetime $\square\phi(x) = 0$. The field stress-energy tensor is quadratic in the field, and independent of $h(x)$. The classical gravitational field equations will be given by ²

$$\square h(x) = \kappa T(x), \quad (1.1)$$

where $T(x)$ is the (scalar) trace of the stress-energy tensor, $T(x) = \partial_a \phi(x) \partial^a \phi(x)$ and $\kappa \equiv 16\pi G$, where G is Newton's constant. Note that this is not a self-consistent theory since $\phi(x)$ does not react to the gravitational field $h(x)$. We should also emphasize that this model is not the standard

²In this article we use the $(-, +, +, +)$ sign conventions of Refs. [51, 52], and units in which $c = \hbar = 1$.

linearized theory of gravity in which T is also linear in $h(x)$. It captures, however, some of the key features of linearized gravity.

In the Heisenberg representation the quantum field $\hat{h}(x)$ satisfies

$$\square \hat{h}(x) = \kappa \hat{T}(x). \quad (1.2)$$

Since $\hat{T}(x)$ is quadratic in the field operator $\hat{\phi}(x)$ some regularization procedure has to be assumed in order for (1.2) to make sense. Since we work in flat spacetime we may simply use a normal ordering prescription to regularize the operator $\hat{T}(x)$. The solutions of this equation, i.e. the field operator at the point x , $\hat{h}(x)$, may be written in terms of the retarded propagator $G(x, y)$ as,

$$\hat{h}(x) = \hat{h}^0(x) + \frac{1}{\kappa} \int dx' G(x, x') \hat{T}(x'), \quad (1.3)$$

where $\hat{h}^0(x)$ is the free field which carries information on the initial conditions and the state of the field. From this solution we may compute, for instance, the symmetric two point quantum correlation function (the anticommutator)

$$\frac{1}{2} \langle \{\hat{h}(x), \hat{h}(y)\} \rangle = \frac{1}{2} \langle \{\hat{h}^0(x), \hat{h}^0(y)\} \rangle + \frac{1}{2\kappa^2} \int \int dx' dy' G(x, x') G(y, y') \langle \{\hat{T}(x'), \hat{T}(y')\} \rangle, \quad (1.4)$$

where the expectation value is taken with respect to the quantum state in which both fields $\phi(x)$ and $h(x)$ are quantized. (We assume for the free field, $\langle \hat{h}^0 \rangle = 0$.)

We can now consider the semiclassical theory for this problem. If we assume that $h(x)$ is classical and the matter field is quantum the semiclassical theory may just be described by substituting into the classical equation (1.1) the stress-energy trace by the expectation value of the stress-energy trace operator $\langle \hat{T}(x) \rangle$, in some quantum state of the field $\hat{\phi}(x)$. Since in our model $\hat{T}(x)$ is independent of $h(x)$ we may simply renormalize its expectation value using normal ordering, then for the vacuum state of the field $\hat{\phi}(x)$, we would simply have $\langle \hat{T}(x) \rangle_0 = 0$. The semiclassical theory thus reduces to

$$\square h(x) = \kappa \langle \hat{T}(x) \rangle. \quad (1.5)$$

The two point function $h(x)h(y)$ that one may derive from this equation depends on the two point function $\langle \hat{T}(x) \rangle \langle \hat{T}(y) \rangle$ and clearly cannot reproduce the quantum result (1.4) which depends on the expectation value of two point operator $\langle \{\hat{T}(x), \hat{T}(y)\} \rangle$. That is, the semiclassical theory entirely misses the fluctuations of the stress-energy operator $\hat{T}(x)$.

Let us now see how we can extend the semiclassical theory in order to account for such fluctuations. The first step is to characterize these fluctuations. For this, we introduce the noise kernel as the physical observable that measures the fluctuations of the stress-energy operator \hat{T} . Define

$$N(x, y) = \frac{1}{2} \langle \{\hat{t}(x), \hat{t}(y)\} \rangle \quad (1.6)$$

where $\hat{t}(x) = \hat{T}(x) - \langle \hat{T}(x) \rangle$. The bi-scalar $N(x, y)$ is real and positive-semidefinite, as a consequence of \hat{t} being self-adjoint. A simple proof can be given as follows. Let $|\psi\rangle$ be a given quantum state and let \hat{Q} be a self-adjoint operator, $\hat{Q}^\dagger = \hat{Q}$, then one can write $\langle \psi | \hat{Q} \hat{Q} | \psi \rangle = \langle \psi | \hat{Q}^\dagger \hat{Q} | \psi \rangle = |\hat{Q}|\psi\rangle|^2 \geq 0$. Now let $\hat{t}(x)$ be a self-adjoint operator, then if we define $\hat{Q} = \int dx f(x) \hat{t}(x)$ for an arbitrary well behaved function $f(x)$, the previous inequality can be written as $\int dx dy f(x) \langle \psi | \hat{t}(x) \hat{t}(y) | \psi \rangle f(y) \geq 0$,

which is the condition for the noise kernel to be positive semi-definite. Note that when considering the inverse kernel $N^{-1}(x, y)$, it is implicitly assumed that one is working in the subspace obtained from the eigenvectors which have strictly positive eigenvalues when the noise kernel is diagonalized.

By the positive semi-definite property of the noise kernel $N(x, y)$ it is possible to introduce a Gaussian stochastic field as follows:

$$\langle \xi(x) \rangle_s = 0, \quad \langle \xi(x)\xi(y) \rangle_s = N(x, y). \quad (1.7)$$

where the subscript s means a statistical average. These equations entirely define the stochastic process $\xi(x)$ since we have assumed that it is Gaussian. Of course, higher correlations could also be introduced but we just try to capture the fluctuations to lowest order.

The extension of the semiclassical equation may be simply performed by adding to the right-hand side of the semiclassical equation (1.5) this stochastic source $\xi(x)$ which accounts for the fluctuations of \hat{T} as follows,

$$\square h(x) = \kappa \left(\langle \hat{T}(x) \rangle + \xi(x) \right). \quad (1.8)$$

This equation is in the form of a Langevin equation: the field $h(x)$ is classical but stochastic and the observables we may obtain from it are correlation functions for $h(x)$. In fact, the solution of this equation may be written in terms of the retarded propagator as,

$$h(x) = h^0(x) + \frac{1}{\kappa} \int dx' G(x, x') \left(\langle \hat{T}(x') \rangle + \xi(x') \right), \quad (1.9)$$

from where the two point correlation function for the classical field $h(x)$, after using the definition of $\xi(x)$ and that $\langle h^0(x) \rangle_s = 0$, is given by

$$\langle h(x)h(y) \rangle_s = \langle h^0(x)h^0(y) \rangle_s + \frac{1}{2\kappa^2} \int dx' dy' G(x, x') G(y, y') \langle \{ \hat{T}(x'), \hat{T}(y') \} \rangle. \quad (1.10)$$

Note that in writing $\langle \dots \rangle_s$ here we are assuming a double stochastic average, one is related to the stochastic field $\xi(x)$ and the other is related to the free field $h^0(x)$ which is assumed also to be stochastic with a distribution function to be specified.

Comparing (1.4) with (1.10) we see that the respective second term on the right-hand side are identical provided the expectation values are computed in the same quantum state for the field $\hat{\phi}(x)$ (recall that we have assumed $T(x)$ does not depend on $h(x)$). The fact that the field $h(x)$ is also quantized in (1.4) does not change the previous statement. The nature of the first term on the right-hand sides of equations (1.4) and (1.10) is different: in the first case it is the two point quantum expectation value of the free quantum field \hat{h}^0 whereas in the second case it is the stochastic average of the two point classical homogeneous field h^0 , which depends on the initial conditions. Now we can still make these terms equal to each other if we assume for the homogeneous field h^0 a Gaussian distribution of initial conditions such that

$$\langle h^0(x)h^0(y) \rangle_s = \frac{1}{2} \langle \{ \hat{h}^0(x), \hat{h}^0(y) \} \rangle. \quad (1.11)$$

This Gaussian stochastic field $h^0(x)$ can always be defined due to the positivity of the anti-commutator. Thus, under this assumption on the initial conditions for the field $h(x)$ the two point correlation function of (1.10) equals the quantum expectation value of (1.4) exactly. An interesting feature of the stochastic description is that the quantum anticommutator of (1.4) can be

written as the right-hand side of equation (1.10) where the first term contains all the information on initial conditions for the stochastic field $h(x)$ and the second term codifies all the information on the quantum correlations of the source. This separation is also seen in the description of some quantum Brownian motion models which are typically used as paradigms of open quantum systems [53, 54].

It is interesting to note that in the standard linearized theory of gravity $T(x)$ depends also on $h(x)$, both explicitly and also implicitly through the coupling of $\phi(x)$ with $h(x)$. The equations are not so simple but it is still true that the corresponding Langevin equation leads to the correct symmetrized two point quantum correlations for the metric perturbations [28, 50]. Thus in a linear theory as in the model just described one may just use the statistical description given by (1.8) to compute the symmetric quantum two point function of equation (1.3). This does not mean that we can recover all quantum correlation functions with the stochastic description, see Ref. [53] for a general discussion about this point. Note that, for instance, the commutator of the classical stochastic field $h(x)$ is obviously zero, but the commutator of the quantum field $\hat{h}(x)$ is not zero for timelike separated points; this is the prize we pay for the introduction of the classical field $\xi(x)$ to describe the quantum fluctuations. Furthermore, the statistical description is not able to account for the graviton-graviton effects which go beyond the linear approximation in $\hat{h}(x)$.

2 The Einstein-Langevin equation: Axiomatic approach

In this section we introduce *stochastic semiclassical gravity* in an axiomatic way, following closely the previous toy model. It is introduced as an extension of semiclassical gravity motivated by the search of self-consistent equations which describe the back-reaction of the quantum stress-energy fluctuations on the gravitational field [49].

2.1 Semiclassical gravity

Semiclassical gravity describes the interaction of a classical gravitational field with quantum matter fields. This theory can be formally derived as the leading $1/N$ approximation of quantum gravity interacting with N independent and identical free quantum fields [55, 56, 57] which interact with gravity only. By keeping the value of NG finite, where G is Newton's gravitational constant, one arrives at a theory in which formally the gravitational field can be treated as a c-number field (i.e. quantized at tree level) while matter fields are fully quantized.

The semiclassical theory may be summarized as follows. Let (\mathcal{M}, g_{ab}) be a globally hyperbolic four-dimensional spacetime manifold \mathcal{M} with metric g_{ab} and consider a real scalar quantum field ϕ of mass m propagating on that manifold; we just assume a scalar field for simplicity. The classical action S_m for this matter field is given by the functional

$$S_m[g, \phi] = -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{ab} \nabla_a \phi \nabla_b \phi + (m^2 + \xi R) \phi^2 \right], \quad (2.1)$$

where ∇_a is the covariant derivative associated to the metric g_{ab} , ξ is a coupling parameter between the field and the scalar curvature of the underlying spacetime R , and $g = \det g_{ab}$.

The field may be quantized in the manifold using the standard canonical quantization formalism [5, 6, 7]. The field operator in the Heisenberg representation $\hat{\phi}$ is an operator-valued distribution

solution of the Klein-Gordon equation, the field equation derived from Eq. (2.1),

$$(\square - m^2 - \xi R)\hat{\phi} = 0. \quad (2.2)$$

We may write the field operator as $\hat{\phi}[g; x]$ to indicate that it is a functional of the metric g_{ab} and a function of the spacetime point x . This notation will be used also for other operators and tensors.

The classical stress-energy tensor is obtained by functional derivation of this action in the usual way $T^{ab}(x) = (2/\sqrt{-g})\delta S_m/\delta g_{ab}$, leading to

$$\begin{aligned} T^{ab}[g, \phi] &= \nabla^a \phi \nabla^b \phi - \frac{1}{2} g^{ab} (\nabla^c \phi \nabla_c \phi + m^2 \phi^2) \\ &+ \xi (g^{ab} \square - \nabla^a \nabla^b + G^{ab}) \phi^2, \end{aligned} \quad (2.3)$$

where $\square = \nabla_a \nabla^a$ and G_{ab} is the Einstein tensor. With the notation $T^{ab}[g, \phi]$ we explicitly indicate that the stress-energy tensor is a functional of the metric g_{ab} and the field ϕ .

The next step is to define a stress-energy tensor operator $\hat{T}^{ab}[g; x]$. Naively one would replace the classical field $\phi[g; x]$ in the above functional by the quantum operator $\hat{\phi}[g; x]$, but this procedure involves taking the product of two distributions at the same spacetime point. This is ill-defined and we need a regularization procedure. There are several regularization methods which one may use, one is the point-splitting or point-separation regularization method [12] in which one introduces a point y in a neighborhood of the point x and then uses as the regulator the vector tangent at the point x of the geodesic joining x and y ; this method is discussed for instance in Ref. [24, 25, 26]. Another well known method is dimensional regularization in which one works in arbitrary n dimensions, where n is not necessarily an integer, and then uses as the regulator the parameter $\epsilon = n - 4$; this method is implicitly used in this section. The regularized stress-energy operator using the Weyl ordering prescription, *i.e.* symmetrical ordering, can be written as

$$\hat{T}^{ab}[g] = \frac{1}{2} \{ \nabla^a \hat{\phi}[g], \nabla^b \hat{\phi}[g] \} + \mathcal{D}^{ab}[g] \hat{\phi}^2[g], \quad (2.4)$$

where $\mathcal{D}^{ab}[g]$ is the differential operator: $\mathcal{D}^{ab} \equiv (\xi - 1/4) g^{ab} \square + \xi (R^{ab} - \nabla^a \nabla^b)$. Note that if dimensional regularization is used, the field operator $\hat{\phi}[g; x]$ propagates in a n -dimensional space-time. Once the regularization prescription has been introduced a regularized and renormalized stress-energy operator $\hat{T}_{ab}^R[g; x]$ may be defined which differs from the regularized $\hat{T}_{ab}[g; x]$ by the identity operator times some tensor counterterms, which depend on the regulator and are local functionals of the metric, see Ref. [27] for details. The field states can be chosen in such a way that for any pair of physically acceptable states, *i.e.*, Hadamard states in the sense of Ref. [7], $|\psi\rangle$, and $|\varphi\rangle$ the matrix element $\langle \psi | T_{ab}^R | \varphi \rangle$, defined as the limit when the regulator takes the physical value, is finite and satisfies Wald's axioms [6, 13]. These counterterms can be extracted from the singular part of a Schwinger-DeWitt series [6, 12, 58]. The choice of these counterterms is not unique but this ambiguity can be absorbed into the renormalized coupling constants which appear in the equations of motion for the gravitational field.

The *semiclassical Einstein equation* for the metric g_{ab} can then be written as

$$G_{ab}[g] + \Lambda g_{ab} - 2(\alpha A_{ab} + \beta B_{ab})[g] = 8\pi G \langle \hat{T}_{ab}^R[g] \rangle, \quad (2.5)$$

where $\langle \hat{T}_{ab}^R[g] \rangle$ is the expectation value of the operator $\hat{T}_{ab}^R[g, x]$ after the regulator takes the physical value in some physically acceptable state of the field on (\mathcal{M}, g_{ab}) . Note that both the stress

tensor and the quantum state are functionals of the metric, hence the notation. The parameters G , Λ , α and β are the renormalized coupling constants, respectively, the gravitational constant, the cosmological constant and two dimensionless coupling constants which are zero in the classical Einstein equation. These constants must be understood as the result of “dressing” the bare constants which appear in the classical action before renormalization. The values of these constants must be determined by experiment. The left hand side of Eq. (2.5) may be derived from the gravitational action

$$S_g[g] = \frac{1}{8\pi G} \int d^4x \sqrt{-g} \left[\frac{1}{2} R - \Lambda + \alpha C_{abcd} C^{abcd} + \beta R^2 \right], \quad (2.6)$$

where C_{abcd} is the Weyl tensor. The tensors A_{ab} and B_{ab} come from the functional derivatives with respect to the metric of the terms quadratic in the curvature in Eq. (2.6), they are explicitly given by

$$\begin{aligned} A^{ab} &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int d^4x \sqrt{-g} C_{cdef} C^{cdef} \\ &= \frac{1}{2} g^{ab} C_{cdef} C^{cdef} - 2 R^{acde} R^b_{cde} + 4 R^{ac} R^b_c - \frac{2}{3} R R^{ab} \\ &\quad - 2 \square R^{ab} + \frac{2}{3} \nabla^a \nabla^b R + \frac{1}{3} g^{ab} \square R, \end{aligned} \quad (2.7)$$

$$\begin{aligned} B^{ab} &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int d^4x \sqrt{-g} R^2 \\ &= \frac{1}{2} g^{ab} R^2 - 2 R R^{ab} + 2 \nabla^a \nabla^b R - 2 g^{ab} \square R, \end{aligned} \quad (2.8)$$

where R_{abcd} and R_{ab} are the Riemann and Ricci tensors, respectively. These two tensors are, like the Einstein and metric tensors, symmetric and divergenceless: $\nabla^a A_{ab} = 0 = \nabla^a B_{ab}$. Note that a classical stress-energy tensor can also be added to the right hand side of Eq. (2.5), but we omit such a term for simplicity.

A solution of semiclassical gravity consists of a spacetime (\mathcal{M}, g_{ab}) , a quantum field operator $\hat{\phi}[g]$ which satisfies the evolution equation (2.2), and a physically acceptable state $|\psi[g]\rangle$ for this field, such that Eq. (2.5) is satisfied when the expectation value of the renormalized stress-energy operator is evaluated in this state.

For a free quantum field this theory is robust in the sense that it is self-consistent and fairly well understood. As long as the gravitational field is assumed to be described by a classical metric, the above semiclassical Einstein equations seems to be the only plausible dynamical equation for this metric: the metric couples to matter fields via the stress-energy tensor and for a given quantum state the only physically observable c-number stress-energy tensor that one can construct is the above renormalized expectation value. However, lacking a full quantum gravity theory the scope and limits of the theory are not so well understood. It is assumed that the semiclassical theory should break down at Planck scales, which is when simple order of magnitude estimates suggest that the quantum effects of gravity should not be ignored because the energy of a quantum fluctuation in a Planck size region, as determined by the Heisenberg uncertainty principle, is comparable to the gravitational energy of that fluctuation.

The theory is expected to break down when the fluctuations of the stress-energy operator are large [19]. A criterion based on the ratio of the fluctuations to the mean was proposed by Kuo and Ford [20] (see also work via zeta-function methods [21, 32]). This proposal was questioned by

Phillips and Hu [22, 23, 24] because it does not contain a scale at which the theory is probed or how accurately the theory can be resolved. They suggested the use of a smearing scale or point-separation distance, for integrating over the bi-tensor quantities, equivalent to a stipulation of the resolution level of measurements. (See response by Ford [33, 34]). A different criterion is recently suggested by Anderson et al [35] based on linear response theory. A partial summary of this issue can be found in our Erice Lectures [2].

2.2 Stochastic gravity

The purpose of stochastic semiclassical gravity is to extend the semiclassical theory to account for these fluctuations in a self-consistent way. A physical observable that describes these fluctuations to lowest order is the *noise kernel* bi-tensor, which is defined through the two point correlation of the stress-energy operator as

$$N_{abcd}[g; x, y) = \frac{1}{2} \langle \{ \hat{t}_{ab}[g; x), \hat{t}_{cd}[g; y) \} \rangle, \quad (2.9)$$

where the curly brackets mean anticommutator, and where

$$\hat{t}_{ab}[g; x) \equiv \hat{T}_{ab}[g; x) - \langle \hat{T}_{ab}[g; x) \rangle. \quad (2.10)$$

This bi-tensor is sometimes also written $N_{abc'd'}[g; x, y)$, or $N_{abc'd'}(x, y)$, to emphasize that it is a tensor with respect to the first two indices at the point x and a tensor with respect to the last two indices at the point y , but we shall not follow this notation here. The noise kernel is defined in terms of the unrenormalized stress-tensor operator $\hat{T}_{ab}[g; x)$ on a given background metric g_{ab} , thus a regulator is implicitly assumed on the right-hand side of Eq. (2.9). However, for a linear quantum field the above kernel – the expectation function of a bi-tensor – is free of ultraviolet divergences because the regularized $T_{ab}[g; x)$ differs from the renormalized $T_{ab}^R[g; x)$ by the identity operator times some tensor counterterms so that in the subtraction (2.10) the counterterms cancel. Consequently the ultraviolet behavior of $\langle \hat{T}_{ab}(x) \hat{T}_{cd}(y) \rangle$ is the same as that of $\langle \hat{T}_{ab}(x) \rangle \langle \hat{T}_{cd}(y) \rangle$, and \hat{T}_{ab} can be replaced by the renormalized operator \hat{T}_{ab}^R in Eq. (2.9); an alternative proof of this result is given in Ref. [24, 25]. The noise kernel should be thought of as a distribution function, the limit of coincidence points has meaning only in the sense of distributions.

The bi-tensor $N_{abcd}[g; x, y)$, or $N_{abcd}(x, y)$ for short, is real and positive semi-definite, as a consequence of \hat{T}_{ab}^R being self-adjoint. A simple proof can be given as in the toy model example discussed above provided one assumes that x in that proof carries also tensorial indices.

Once the fluctuations of the stress-energy operator have been characterized we can perturbatively extend the semiclassical theory to account for such fluctuations. Thus we will assume that the background spacetime metric g_{ab} is a solution of the semiclassical Einstein Eqs. (2.5) and we will write the new metric for the extended theory as $g_{ab} + h_{ab}$, where we will assume that h_{ab} is a perturbation to the background solution. The renormalized stress-energy operator and the state of the quantum field may now be denoted by $\hat{T}_{ab}^R[g + h]$ and $|\psi[g + h]\rangle$, respectively, and $\langle \hat{T}_{ab}^R[g + h] \rangle$ will be the corresponding expectation value.

Let us now introduce a Gaussian stochastic tensor field $\xi_{ab}[g; x)$ defined by the following correlators:

$$\langle \xi_{ab}[g; x) \rangle_s = 0, \quad \langle \xi_{ab}[g; x) \xi_{cd}[g; y) \rangle_s = N_{abcd}[g; x, y), \quad (2.11)$$

where $\langle \dots \rangle_s$ means statistical average. The symmetry and positive semi-definite property of the noise kernel guarantees that the stochastic field tensor $\xi_{ab}[g, x]$, or $\xi_{ab}(x)$ for short, just introduced is well defined. Note that this stochastic tensor captures only partially the quantum nature of the fluctuations of the stress-energy operator since it assumes that cumulants of higher order are zero.

An important property of this stochastic tensor is that it is covariantly conserved in the background spacetime $\nabla^a \xi_{ab}[g; x] = 0$. In fact, as a consequence of the conservation of $\hat{T}_{ab}^R[g]$ one can see that $\nabla_x^a N_{abcd}(x, y) = 0$. Taking the divergence in Eq. (2.11) one can then show that $\langle \nabla^a \xi_{ab} \rangle_s = 0$ and $\langle \nabla_x^a \xi_{ab}(x) \xi_{cd}(y) \rangle_s = 0$ so that $\nabla^a \xi_{ab}$ is deterministic and represents with certainty the zero vector field in \mathcal{M} .

For a conformal field, *i.e.*, a field whose classical action is conformally invariant, ξ_{ab} is traceless: $g^{ab} \xi_{ab}[g; x] = 0$; so that, for a conformal matter field the stochastic source gives no correction to the trace anomaly. In fact, from the trace anomaly result which states that $g^{ab} \hat{T}_{ab}^R[g]$ is in this case a local c-number functional of g_{ab} times the identity operator, we have that $g^{ab}(x) N_{abcd}[g; x, y] = 0$. It then follows from Eq. (2.11) that $\langle g^{ab} \xi_{ab} \rangle_s = 0$ and $\langle g^{ab}(x) \xi_{ab}(x) \xi_{cd}(y) \rangle_s = 0$; an alternative proof based on the point-separation method is given in Ref. [24, 25, 26], see also Ref. [2].

All these properties make it quite natural to incorporate into the Einstein equations the stress-energy fluctuations by using the stochastic tensor $\xi_{ab}[g; x]$ as the source of the metric perturbations. Thus we will write the following equation.

$$G_{ab}[g + h] + \Lambda(g_{ab} + h_{ab}) - 2(\alpha A_{ab} + \beta B_{ab})[g + h] = 8\pi G \left(\langle \hat{T}_{ab}^R[g + h] \rangle + \xi_{ab}[g] \right). \quad (2.12)$$

This equation is in the form of a (*semiclassical*) *Einstein-Langevin equation*, it is a dynamical equation for the metric perturbation h_{ab} to linear order. It describes the back-reaction of the metric to the quantum fluctuations of the stress-energy tensor of matter fields, and gives a first order extension to semiclassical gravity as described by the semiclassical Einstein equation (2.5).³ The renormalization of the operator $\hat{T}_{ab}[g + h]$ is carried out exactly as in the previous case, now in the perturbed metric $g_{ab} + h_{ab}$. Note that the stochastic source $\xi_{ab}[g; x]$ is not dynamical, it is independent of h_{ab} since it describes the fluctuations of the stress tensor on the semiclassical background g_{ab} .

An important property of the Einstein-Langevin equation is that it is gauge invariant under the change of h_{ab} by $h'_{ab} = h_{ab} + \nabla_a \zeta_b + \nabla_b \zeta_a$, where ζ^a is a stochastic vector field on the background manifold \mathcal{M} . Note that a tensor such as $R_{ab}[g + h]$, transforms as $R_{ab}[g + h'] = R_{ab}[g + h] + \mathcal{L}_\zeta R_{ab}[g]$ to linear order in the perturbations, where \mathcal{L}_ζ is the Lie derivative with respect to ζ^a . Now, let us write the source tensors in Eqs. (2.12) and (2.5) to the left-hand sides of these equations. If we

³Note that we refer to the ELE as a first order extension to semiclassical Einstein equation (SEE) of semiclassical gravity (SCG) and the lowest level representation of stochastic gravity because in practice the Feynman-Vernon scheme of representing the noise kernel by a classical stochastic source is implementable only in cases where that term in the influence action contains quadratures of the system variable h [see Eq. (3.9)] via a Gaussian functional integral identity [43]. This is possible for linear perturbations but not necessarily so at higher orders, because of the difficulty in rendering it to a form amenable to the employ of the Feynman-Vernon identity for the definition of a classical stochastic source. However, stochastic gravity has a much broader meaning beyond these limitations. It refers to the range of theories based on second and higher order correlation functions. Noise can be defined in effectively open systems (e.g. correlation noise [59] in the Schwinger-Dyson equation hierarchy) to some degree but one should not expect the simple Langevin form to prevail. It is in this sense that we say stochastic gravity is the intermediate theory between SCG (a mean field theory based on the expectation values of the energy momentum tensor of quantum fields) and quantum gravity (the full hierarchy of correlation functions retaining complete quantum coherence [1, 60])

substitute h by h' in this new version of Eq. (2.12), we get the same expression, with h instead of h' , plus the Lie derivative of the combination of tensors which appear on the left-hand side of the new Eq. (2.5). This last combination vanishes when Eq. (2.5) is satisfied, *i.e.*, when the background metric g_{ab} is a solution of semiclassical gravity.

A solution of Eq. (2.12) can be formally written as $h_{ab}[\xi]$. This solution is characterized by the whole family of its correlation functions. From the statistical average of this equation we have that $g_{ab} + \langle h_{ab} \rangle_s$ must be a solution of the semiclassical Einstein equation linearized around the background g_{ab} ; this solution has been proposed as a test for the validity of the semiclassical approximation [35]. The fluctuation of the metric around this average are described by the moments of the stochastic field $h_{ab}^s[\xi] = h_{ab}[\xi] - \langle h_{ab} \rangle_s$. Thus the solutions of the Einstein-Langevin equation will provide the two point metric correlators $\langle h_{ab}^s(x) h_{cd}^s(y) \rangle_s$.

We see that whereas the semiclassical theory depends on the expectation value of the point-defined value of the stress-energy operator, the stochastic theory carries information also on the two point correlation of the stress-energy operator, as is shown in the toy model of the previous section, the stochastic theory may be understood as an intermediate step between the semiclassical theory based on the mean value, and the full quantum theory in the sense that it contains extra, yet only partial, information carried by the n-point functions.

We should also emphasize that, even if the metric fluctuations appears classical and stochastic, their origin is quantum not only because they are induced by the fluctuations of quantum matter, but also because they are the suitably coarse-grained variables left over from the quantum gravity fluctuations after some mechanism for decoherence and classicalization of the metric field [45, 61, 62, 63, 64]. One may, in fact, derive the stochastic semiclassical theory from a full quantum theory. This was done via the world-line influence functional method for a moving charged particle in an electromagnetic field in quantum electrodynamics [65]. From another viewpoint, quite independent of whether a classicalization mechanism is mandatory or implementable, the Einstein-Langevin equation proves to be a useful tool to compute the symmetrized two point correlations of the quantum metric perturbations [50], as illustrated in the simple toy model described previously.

3 The Einstein-Langevin equation: Functional approach

The Einstein-Langevin equation (2.12) may also be derived by a method based on functional techniques [27]. In semiclassical gravity functional methods were used to study the back-reaction of quantum fields in cosmological models [16]. The primary advantage of the effective action approach is, in addition to the well-known fact that it is easy to introduce perturbation schemes like loop expansion and nPI formalisms, that it yields a *fully* self-consistent solution.⁴ The well known in-out effective action method treated in textbooks, however, led to equations of motion which were not real because they were tailored to compute transition elements of quantum operators

⁴This is because the set of equations of motion derived from the effective action for *both* the spacetime and the quantum fields are of the same order. Note that seeking backreaction from the equations of motion level *ab initio* (not derived from the effective action) straightforwardly, as done in the old ways, *i.e.*, proceeding with the solution of one *after* another would produce results in staggered orders. To compare these two approaches, equation of motion versus effective action, see, e.g., the work of Hu and Parker 1978 versus Hartle and Hu 1979 in [15]. See also comments in Sec. 5.6 on the black hole backreaction problem comparing the approach by York et al [83] versus that of Sinha, Raval and Hu [116]. The correct way at the equation of motion level for stochastic gravity is that illustrated in the previous section.

rather than expectation values. The correct technique to use for the backreaction problem is the Schwinger-Keldysh [66] closed-time-path (CTP) or ‘in-in’ effective action. These techniques were adapted to the gravitational context [67, 68] and applied to different problems in cosmology. One could deduce the semiclassical Einstein equation from the CTP effective action for the gravitational field (at tree level) with quantum matter fields.

Furthermore, in this case the CTP functional formalism turns out to be related [69, 38, 41, 70, 71, 72, 73, 27] to the influence functional formalism of Feynman and Vernon [43] since the full quantum system may be understood as consisting of a distinguished subsystem (the “system” of interest) interacting with the remaining degrees of freedom (the environment). Integration out the environment variables in a CTP path integral yields the influence functional, from which one can define an effective action for the dynamics of the system [38, 40, 74, 71]. This approach to semiclassical gravity is motivated by the observation [37] that in some open quantum systems classicalization and decoherence [46] on the system may be brought about by interaction with an environment, the environment being in this case the matter fields and some “high-momentum” gravitational modes [75, 64]. Unfortunately, since the form of a complete quantum theory of gravity interacting with matter is unknown, we do not know what these “high-momentum” gravitational modes are. Such a fundamental quantum theory might not even be a field theory, in which case the metric and scalar fields would not be fundamental objects [1]. Thus, in this case, we cannot attempt to evaluate the influence action of Feynman and Vernon starting from the fundamental quantum theory and performing the path integrations in the environment variables. Instead, we introduce the influence action for an effective quantum field theory of gravity and matter [76, 77, 48], in which such “high-momentum” gravitational modes are assumed to have already been “integrated out.”

Adopting the usual procedure of effective field theories [79, 76, 78], one has to take the effective action for the metric and the scalar field of the most general local form compatible with general covariance: $S[g, \phi] \equiv S_g[g] + S_m[g, \phi] + \dots$, where $S_g[g]$ and $S_m[g, \phi]$ are given by Eqs. (2.6) and (2.1), respectively, and the dots stand for terms of order higher than two in the curvature and in the number of derivatives of the scalar field. Here, we shall neglect the higher order terms as well as self-interaction terms for the scalar field. The second order terms are necessary to renormalize one-loop ultraviolet divergences of the scalar field stress-energy tensor, as we have already seen. Since \mathcal{M} is a globally hyperbolic manifold, we can foliate it by a family of $t = \text{constant}$ Cauchy hypersurfaces Σ_t , and we will indicate the initial and final times by t_i and t_f , respectively.

The *influence functional* corresponding to the action (2.1) describing a scalar field in a spacetime (coupled to a metric field) may be introduced as a functional of two copies of the metric, denoted by g_{ab}^+ and g_{ab}^- , which coincide at some final time $t = t_f$. Let us assume that, in the quantum effective theory, the state of the full system (the scalar and the metric fields) in the Schrödinger picture at the initial time $t = t_i$ can be described by a density operator which can be written as the tensor product of two operators on the Hilbert spaces of the metric and of the scalar field. Let $\rho_i[\phi_+(t_i), \phi_-(t_i)]$ be the matrix element of the density operator $\hat{\rho}^S(t_i)$ describing the initial state of the scalar field. The Feynman-Vernon influence functional is defined as the following path integral over the two copies of the scalar field:

$$\mathcal{F}_{\text{IF}}[g^\pm] \equiv \int \mathcal{D}\phi_+ \mathcal{D}\phi_- \rho_i[\phi_+(t_i), \phi_-(t_i)] \delta[\phi_+(t_f) - \phi_-(t_f)] e^{i(S_m[g^+, \phi_+] - S_m[g^-, \phi_-])}. \quad (3.1)$$

Alternatively, the above double path integral can be rewritten as a closed time path (CTP) integral, namely, as a single path integral in a complex time contour with two different time branches, one

going forward in time from t_i to t_f , and the other going backward in time from t_f to t_i (in practice one usually takes $t_i \rightarrow -\infty$). From this influence functional, the *influence action* $S_{\text{IF}}[g^+, g^-]$, or $S_{\text{IF}}[g^\pm]$ for short, defined by

$$\mathcal{F}_{\text{IF}}[g^\pm] \equiv e^{iS_{\text{IF}}[g^\pm]}, \quad (3.2)$$

carries all the information about the environment (the matter fields) relevant to the system (the gravitational field). Then we can define the CTP *effective action* for the gravitational field, $S_{\text{eff}}[g^\pm]$, as

$$S_{\text{eff}}[g^\pm] \equiv S_g[g^+] - S_g[g^-] + S_{\text{IF}}[g^\pm]. \quad (3.3)$$

This is the effective action for the classical gravitational field in the CTP formalism. However, since the gravitational field is treated only at the tree level, this is also the effective classical action from which the classical equations of motion can be derived.

Expression (3.1) contains ultraviolet divergences and must be regularized. We shall assume that dimensional regularization can be applied, that is, it makes sense to dimensionally continue all the quantities that appear in Eq. (3.1). For this we need to work with the n -dimensional actions corresponding to S_m in (3.1) and S_g in (2.6). For example, the parameters G , Λ , α and β of Eq. (2.6) are the bare parameters G_B , Λ_B , α_B and β_B , and in $S_g[g]$, instead of the square of the Weyl tensor in Eq. (2.6), one must use $(2/3)(R_{abcd}R^{abcd} - R_{ab}R^{ab})$ which by the Gauss-Bonnet theorem leads to the same equations of motion as the action (2.6) when $n=4$. The form of S_g in n dimensions is suggested by the Schwinger-DeWitt analysis of the ultraviolet divergences in the matter stress-energy tensor using dimensional regularization. One can then write the Feynman-Vernon effective action $S_{\text{eff}}[g^\pm]$ in Eq. (3.3) in a form suitable for dimensional regularization. Since both S_m and S_g contain second order derivatives of the metric, one should also add some boundary terms [52, 40]. The effect of these terms is to cancel out the boundary terms which appear when taking variations of $S_{\text{eff}}[g^\pm]$ keeping the value of g_{ab}^+ and g_{ab}^- fixed at Σ_{t_i} and Σ_{t_f} . Alternatively, in order to obtain the equations of motion for the metric in the semiclassical regime, we can work with the action terms without boundary terms and neglect all boundary terms when taking variations with respect to g_{ab}^\pm . From now on, all the functional derivatives with respect to the metric will be understood in this sense.

The semiclassical Einstein equation (2.5) can now be derived. Using the definition of the stress-energy tensor $T^{ab}(x) = (2/\sqrt{-g})\delta S_m/\delta g_{ab}$ and the definition of the influence functional, Eqs. (3.1) and (3.2), we see that

$$\langle \hat{T}^{ab}[g; x] \rangle = \frac{2}{\sqrt{-g(x)}} \left. \frac{\delta S_{\text{IF}}[g^\pm]}{\delta g_{ab}^\pm(x)} \right|_{g^\pm=g}, \quad (3.4)$$

where the expectation value is taken in the n -dimensional spacetime generalization of the state described by $\hat{\rho}^s(t_i)$. Therefore, differentiating $S_{\text{eff}}[g^\pm]$ in Eq. (3.3) with respect to g_{ab}^\pm , and then setting $g_{ab}^+ = g_{ab}^- = g_{ab}$, we get the semiclassical Einstein equation in n dimensions. This equation is then renormalized by absorbing the divergences in the regularized $\langle \hat{T}^{ab}[g] \rangle$ into the bare parameters. Taking the limit $n \rightarrow 4$ we obtain the physical semiclassical Einstein equation (2.5).

In the spirit of the previous derivation of the Einstein-Langevin equation, we now seek a dynamical equation for a linear perturbation h_{ab} to the semiclassical metric g_{ab} , solution of Eq. (2.5). Strictly speaking if we use dimensional regularization we must consider the n -dimensional version of that equation; see Ref. [27] for details. From the results just described, if such an equation were simply a linearized semiclassical Einstein equation, it could be obtained from an expansion of the

effective action $S_{\text{eff}}[g + h^\pm]$. In particular, since, from Eq. (3.4), we have that

$$\langle \hat{T}^{ab}[g + h; x] \rangle = \frac{2}{\sqrt{-\det(g+h)(x)}} \frac{\delta S_{\text{IF}}[g + h^\pm]}{\delta h_{ab}^\pm(x)} \Big|_{h^\pm=h}, \quad (3.5)$$

the expansion of $\langle \hat{T}^{ab}[g+h] \rangle$ to linear order in h_{ab} can be obtained from an expansion of the influence action $S_{\text{IF}}[g + h^\pm]$ up to second order in h_{ab}^\pm .

To perform the expansion of the influence action, we have to compute the first and second order functional derivatives of $S_{\text{IF}}[g + h^\pm]$ and then set $h_{ab}^+ = h_{ab}^- = h_{ab}$. If we do so using the path integral representation (3.1), we can interpret these derivatives as expectation values of operators. The relevant second order derivatives are

$$\begin{aligned} \frac{4}{\sqrt{-g(x)}\sqrt{-g(y)}} \frac{\delta^2 S_{\text{IF}}[g + h^\pm]}{\delta h_{ab}^+(x)\delta h_{cd}^+(y)} \Big|_{h^\pm=h} &= -H_{\text{S}}^{abcd}[g; x, y] - K^{abcd}[g; x, y] + iN^{abcd}[g; x, y], \\ \frac{4}{\sqrt{-g(x)}\sqrt{-g(y)}} \frac{\delta^2 S_{\text{IF}}[g^\pm]}{\delta h_{ab}^+(x)\delta h_{cd}^-(y)} \Big|_{h^\pm=h} &= -H_{\text{A}}^{abcd}[g; x, y] - iN^{abcd}[g; x, y], \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} N^{abcd}[g; x, y] &\equiv \frac{1}{2} \langle \{ \hat{t}^{ab}[g; x], \hat{t}^{cd}[g; y] \} \rangle, \quad H_{\text{S}}^{abcd}[g; x, y] \equiv \text{Im} \langle \text{T}^* (\hat{T}^{ab}[g; x] \hat{T}^{cd}[g; y]) \rangle, \\ H_{\text{A}}^{abcd}[g; x, y] &\equiv -\frac{i}{2} \langle [\hat{T}^{ab}[g; x], \hat{T}^{cd}[g; y]] \rangle, \quad K^{abcd}[g; x, y] \equiv \frac{-4}{\sqrt{-g(x)}\sqrt{-g(y)}} \left\langle \frac{\delta^2 S_m[g + h, \phi]}{\delta h_{ab}(x)\delta h_{cd}(y)} \Big|_{\phi=\hat{\phi}} \right\rangle, \end{aligned}$$

with \hat{t}^{ab} defined in Eq. (2.10), $[,]$ denotes the commutator and $\{ , \}$ the anti-commutator. Here we use a Weyl ordering prescription for the operators in the last of these expressions the symbol T^* to denote the following ordered operations: First, time order the field operators $\hat{\phi}$ and then apply the derivative operators which appear in each term of the product $T^{ab}(x)T^{cd}(y)$, where T^{ab} is the functional (2.3). This T^* “time ordering” arises because we have path integrals containing products of derivatives of the field, which can be expressed as derivatives of the path integrals which do not contain such derivatives. Notice, from their definitions, that all the kernels which appear in expressions (3.6) are real and also H_{A}^{abcd} is free of ultraviolet divergences in the limit $n \rightarrow 4$.

From (3.4) and (3.6), taking into account that $S_{\text{IF}}[g, g] = 0$ and that $S_{\text{IF}}[g^-, g^+] = -S_{\text{IF}}^*[g^+, g^-]$, we can write the expansion for the influence action $S_{\text{IF}}[g + h^\pm]$ around a background metric g_{ab} in terms of the previous kernels. Taking into account that these kernels satisfy the symmetry relations

$$H_{\text{S}}^{abcd}(x, y) = H_{\text{S}}^{cdab}(y, x), \quad H_{\text{A}}^{abcd}(x, y) = -H_{\text{A}}^{cdab}(y, x), \quad K^{abcd}(x, y) = K^{cdab}(y, x), \quad (3.7)$$

and introducing the new kernel

$$H^{abcd}(x, y) \equiv H_{\text{S}}^{abcd}(x, y) + H_{\text{A}}^{abcd}(x, y), \quad (3.8)$$

the expansion of S_{IF} can be finally written as

$$\begin{aligned} S_{\text{IF}}[g + h^\pm] &= \frac{1}{2} \int d^4x \sqrt{-g(x)} \langle \hat{T}^{ab}[g; x] \rangle [h_{ab}(x)] \\ &\quad - \frac{1}{8} \int d^4x d^4y \sqrt{-g(x)}\sqrt{-g(y)} [h_{ab}(x)] \left(H^{abcd}[g; x, y] + K^{abcd}[g; x, y] \right) \{h_{cd}(y)\} \\ &\quad + \frac{i}{8} \int d^4x d^4y \sqrt{-g(x)}\sqrt{-g(y)} [h_{ab}(x)] N^{abcd}[g; x, y] [h_{cd}(y)] + 0(h^3), \end{aligned} \quad (3.9)$$

where we have used the notation

$$[h_{ab}] \equiv h_{ab}^+ - h_{ab}^-, \quad \{h_{ab}\} \equiv h_{ab}^+ + h_{ab}^-. \quad (3.10)$$

From Eqs. (3.9) and (3.5) it is clear that the imaginary part of the influence action does not contribute to the perturbed semiclassical Einstein equation (the expectation value of the stress-energy tensor is real), however, as it depends on the noise kernel, it contains information on the fluctuations of the operator $\hat{T}^{ab}[g]$.

We are now in a position to carry out the derivation of the semiclassical Einstein-Langevin equation. The procedure is well known [38, 40, 41, 80]: it consists of deriving a new “stochastic” effective action from the observation that the effect of the imaginary part of the influence action (3.9) on the corresponding influence functional is equivalent to the averaged effect of the stochastic source ξ^{ab} coupled linearly to the perturbations h_{ab}^\pm . This observation follows from the identity first invoked by Feynman and Vernon for such purpose:

$$e^{-\frac{1}{8} \int d^4x d^4y \sqrt{-g(x)} \sqrt{-g(y)} [h_{ab}(x)] N^{abcd}(x, y) [h_{cd}(y)]} = \int \mathcal{D}\xi \mathcal{P}[\xi] e^{\frac{i}{2} \int d^4x \sqrt{-g(x)} \xi^{ab}(x) [h_{ab}(x)]}, \quad (3.11)$$

where $\mathcal{P}[\xi]$ is the probability distribution functional of a Gaussian stochastic tensor ξ^{ab} characterized by the correlators (2.11) with N^{abcd} given by Eq. (2.9), and where the path integration measure is assumed to be a scalar under diffeomorphisms of (\mathcal{M}, g_{ab}) . The above identity follows from the identification of the right-hand side of (3.11) with the characteristic functional for the stochastic field ξ^{ab} . The probability distribution functional for ξ^{ab} is explicitly given by

$$\mathcal{P}[\xi] = \det(2\pi N)^{-1/2} \exp \left[-\frac{1}{2} \int d^4x d^4y \sqrt{-g(x)} \sqrt{-g(y)} \xi^{ab}(x) N_{abcd}^{-1}(x, y) \xi^{cd}(y) \right]. \quad (3.12)$$

We may now introduce the *stochastic effective action* as

$$S_{\text{eff}}^s[g + h^\pm, \xi] \equiv S_g[g + h^+] - S_g[g + h^-] + S_{\text{IF}}^s[g + h^\pm, \xi], \quad (3.13)$$

where the “stochastic” influence action is defined as

$$S_{\text{IF}}^s[g + h^\pm, \xi] \equiv \text{Re } S_{\text{IF}}[g + h^\pm] + \frac{1}{2} \int d^4x \sqrt{-g(x)} \xi^{ab}(x) [h_{ab}(x)] + O(h^3). \quad (3.14)$$

Note that, in fact, the influence functional can now be written as a statistical average over ξ^{ab} : $\mathcal{F}_{\text{IF}}[g + h^\pm] = \langle \exp(iS_{\text{IF}}^s[g + h^\pm, \xi]) \rangle_s$. The stochastic equation of motion for h_{ab} reads

$$\left. \frac{\delta S_{\text{eff}}^s[g + h^\pm, \xi]}{\delta h_{ab}^+(x)} \right|_{h^\pm = h} = 0, \quad (3.15)$$

which is the Einstein-Langevin equation (2.12); notice that only the real part of S_{IF} contributes to the expectation value (3.5). To be precise, we get the regularized n -dimensional equations with the bare parameters, and after renormalization we take the limit $n \rightarrow 4$ to obtain the Einstein-Langevin equation in physical spacetime.

Before ending this section let us consider the causality property of equation (3.15). On general grounds causality is guaranteed from the properties of the expectation values of renormalized stress-energy tensor operators [7]. It is illustrative, however, to check it explicitly in this case. First, we

note that in the expression (3.9) of the S_{IF} only terms which involve the kernels H^{abcd} and K^{abcd} may contain problems concerning causality. The kernel K^{abcd} is local and need not concern us. We want to show that the nonlocal kernel H^{abcd} leads to causal equations, as can be seen from its structure. To simplify the proof let us assume a quantum mechanical operator $\hat{q}(t)$, instead of the stress-energy field operator [29], and suppress tensorial indices, so that in this simplified problem the corresponding kernel will be written as $H(t, t')$. Its general structure, as follows from the definitions in (3.6), is $H(t, t') = H_S(t, t') + H_A(t, t')$, where $H_S(t, t') = \text{Im}\langle T(\hat{q}(t)\hat{q}(t')) \rangle$, with T denoting time ordering, and $H_A(t, t') = -(i/2)\langle [\hat{q}(t), \hat{q}(t')] \rangle$. From this we have that $H(t, t') = -i\langle [\hat{q}(t), \hat{q}(t')] \rangle \theta(t - t')$. Since $H(t, t')$ appears in the equation of motion at time t in a term such as $\int dt' H(t, t') h(t')$, where $h(t')$ plays the role of the metric perturbation, it is clear that the nonlocal term depends on $h(t')$ for times $t' < t$ only so that causality is guaranteed. When we go to the stress-energy tensor field operator the proof is essentially the same [30]: since $\hat{T}^{ab}(x)$ involves spacetime derivatives acting on $\hat{\phi}(x)$ the corresponding θ functions in some of the terms will also carry derivatives but these lead to delta functions which do not destroy the causality property. Furthermore, time ordering is well defined in our spacetime manifold, which is assumed to be globally hyperbolic and thus time orientable.

APPLICATIONS: Backreaction of Particle Creation

Particle creation from a strong or time-dependent gravitational field as in cosmological [9] and black hole [10, 11] spacetimes was studied in the late 60's to the 70's. The effects of particle creation, vacuum polarization and other quantum processes back acting on the background spacetime constitutes what is known as the backreaction problem. Backreaction become increasingly important when the energy reaches the Planck scale, as in the early universe [15] and at the final stages of black hole evolution [81, 82, 83]. Investigation of backreaction problems started with the study of regularization of the stress energy tensor in curved spacetimes in the mid-70's, followed by in-depth calculations of backreaction of particle creation in cosmological spacetimes, such as possible removal of the cosmological singularity by the trace anomaly, and the damping of anisotropy in Bianchi universes by particle creation.

The most significant developments in the implementation and physical aspects of the backreaction problems exemplified in cosmological spacetimes in the 80's are perhaps the introduction of an effective action which yields real and causal equations of motion. (For the axiomatic theoretical aspects, see, e.g., Kay and Wald, Flanagan and Wald [84, 85].) This is known as the closed-time-path (CTP) or the Schwinger-Keldysh method [66]. From this one can identify dissipative effects in an unambiguous manner and in the true statistical mechanical sense. In the 90's, the most significant development was perhaps the introduction of quantum open systems concepts [86] and the influence functional method [43, 87], which enables one to identify the origin of quantum noise and recognize the importance of fluctuations. We give an example of the calculation of the backreaction of cosmological particle creation to illustrate the use of these methods which rest at the base of stochastic gravity theory.

4 Cosmological Backreaction Problems

As a canonical example of cosmological backreaction we consider in this section a massless conformally coupled quantum scalar field on a weakly perturbed spatially flat Friedmann-Robertson-Walker (FRW) spacetime and derive the semiclassical Einstein-Langevin equation for the metric perturbations off this spacetime. These equations were obtained in Ref. [41] following the derivation of the CTP effective action for this problem in Ref. [68]. Einstein-Langevin equations had been previously derived in Ref. [40] for small anisotropies conformally coupled to massless fields in a spatially homogeneous background working in the framework of quantum cosmology, and in Ref. [39] for the scale factor in a spatially flat universe due to the coupling of different quantum scalar fields. The connection between the CTP effective action and the influence functional had been noticed in Ref. [38] in the semiclassical gravity context.

To derive the Einstein-Langevin equation we can compute the influence action S_{IF} , defined in Eq. (3.9), which is equivalent to evaluating the different kernels introduced after Eq. (3.6). This can be done directly from the expressions for the kernels in terms of products of the Feynman and the Wightman propagators for the scalar field in the background metric which were derived in Refs. [27] and [28], or it can be done by an explicit evaluation of the path integrals which define the influence action in Eqs. (3.1) and (3.2). We follow this second, more direct, route in this section.

The metric of our spacetime is given by

$$\tilde{g}_{\mu\nu}(x) = a^2(\eta) (\eta_{\mu\nu} + h_{\mu\nu}(x)), \quad (4.1)$$

where $\eta_{\mu\nu} = \text{diag}(-1, +1, \dots, +1)$, $a(\eta) \equiv e^{\omega(\eta)}$ is the scale factor, η is the conformal time which is related to the cosmological time t by $dt = a d\eta$, $h_{\mu\nu}(x)$ is a symmetric tensor which represents arbitrary small metric perturbations, and we work in an n dimensional spacetime in preparation for dimensional regularization.

The classical action for a free massless conformally coupled real scalar field $\Phi(x)$ is given in n -dimensions by

$$S_m[\tilde{g}_{\mu\nu}, \Phi] = -\frac{1}{2} \int d^n x \sqrt{-\tilde{g}} \left[\tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \xi(n) \tilde{R} \Phi^2 \right], \quad (4.2)$$

where $\xi(n) = (n-2)/[4(n-1)]$, and \tilde{R} is the Ricci scalar for the metric $\tilde{g}_{\mu\nu}$. Because of the conformal coupling $\xi(n)$ we can define a new field $\phi(x)$ and a new metric $g_{\mu\nu}$ as

$$\phi(x) \equiv e^{(n/2-1)\omega(\eta)} \Phi(x), \quad g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + h_{\mu\nu}(x), \quad (4.3)$$

so that $g_{\mu\nu}$ is conformally related to $\tilde{g}_{\mu\nu}$. After one integration by parts the classical action (4.2) takes the form

$$S_m[g_{\mu\nu}, \phi] = -\frac{1}{2} \int d^n x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi(n) R \phi^2 \right] \quad (4.4)$$

which is the action for a free massless conformally coupled real scalar field $\phi(x)$ in a spacetime with metric $g_{\mu\nu}$, *i.e.* a nearly flat spacetime. Although the physical field is $\Phi(x)$ the fact that it is related to the field $\phi(x)$ by a power of the conformal factor implies that a positive frequency mode of the field $\phi(x)$ in flat spacetime will correspond to a positive frequency mode in the conformally related space; these modes define the conformal vacuum. Thus quantum effects such as particle creation will be due to the breaking of conformal flatness which, in this case, is produced by the perturbations

$h_{\mu\nu}(x)$. Expanding the above action in terms of these perturbations, after integrations by parts we have

$$S_m[g_{\mu\nu}, \phi] = \frac{1}{2} \int d^n x \phi [\square + V^{(1)} + V^{(2)} + \dots] \phi, \quad (4.5)$$

where, the operators $V^{(1)}$ and $V^{(2)}$ are defined as

$$\begin{aligned} V^{(1)}(x) &= -\bar{h}^{\mu\nu} \partial_\mu \partial_\nu - (\partial_\mu \bar{h}^{\mu\nu}) \partial_\nu - \xi(n) R^{(1)}, \\ V^{(2)}(x) &= \hat{h}^{\mu\nu} \partial_\mu \partial_\nu + (\partial_\mu \hat{h}^{\mu\nu}) \partial_\nu - \xi(n) \left(R^{(2)} + \frac{1}{2} h R^{(1)} \right), \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} \bar{h}_{\mu\nu} &\equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}, \\ \hat{h}_{\mu\nu} &\equiv h_\mu{}^\alpha h_{\alpha\nu} - \frac{1}{2} h h_{\mu\nu} + \frac{1}{8} h^2 \eta_{\mu\nu} - \frac{1}{4} h_{\alpha\beta} h^{\alpha\beta} \eta_{\mu\nu}, \end{aligned} \quad (4.7)$$

and $R^{(1)}$ and $R^{(2)}$ are the first and second order terms, respectively, in the metric perturbations of the scalar curvature.

To the classical action for the matter fields S_m we have to add the action of the physical metric $\tilde{g}_{\mu\nu}$: $S_g[\tilde{g}_{\mu\nu}]$. As it was emphasized in Sec. 2 in order to renormalize the effective action we need to add appropriate terms quadratic in the Riemann tensor. In this case, using dimensional regularization the only terms needed for renormalization are:

$$\begin{aligned} S_g[\tilde{g}_{\mu\nu}] &\equiv \int d^n x \sqrt{-\tilde{g}(x)} \left\{ \frac{1}{16\pi G_N} \tilde{R}(x) \right. \\ &\quad \left. + \frac{\mu^{n-4}}{2880\pi^2(n-4)} \left[\tilde{R}_{\mu\nu\alpha\beta}(x) \tilde{R}^{\mu\nu\alpha\beta}(x) - \tilde{R}_{\mu\nu}(x) \tilde{R}^{\mu\nu}(x) \right] \right\}. \end{aligned} \quad (4.8)$$

where G_N denotes Newton's gravitational constant and μ is a mass renormalization parameter. Note that in 4-dimensions $\tilde{R}_{\mu\nu\alpha\beta}(x) \tilde{R}^{\mu\nu\alpha\beta}(x) - \tilde{R}_{\mu\nu}(x) \tilde{R}^{\mu\nu}(x) = (3/2) \tilde{C}_{\mu\nu\alpha\beta}(x) \tilde{C}^{\mu\nu\alpha\beta}(x)$ but this is not true when $n \neq 4$, for that reason we have to use the previous combination of Riemann and Ricci tensors instead of the Weyl tensor; furthermore we can add a term proportional to $\tilde{R}^2(x)$ in the gravitational action with an arbitrary coefficient, but since this term is not needed for renormalization we do not introduce it here.

To compute the influence action S_{IF} from Eqs. (3.1) and (3.2) we have to introduce two scalar fields $\phi_+(x)$ and $\phi_-(x)$ which coincide at some future time t_f , $\phi_+(t_f) = \phi_-(t_f)$, and which evolve in two different geometries given by $h_{\mu\nu}^+$ and $h_{\mu\nu}^-$ such that $h_{\mu\nu}^+(t_f) = h_{\mu\nu}^-(t_f)$. The standard Gaussian path integral computation leads to

$$S_{\text{IF}}[h_{\mu\nu}^\pm] = -\frac{i}{2} \text{Tr}(\ln G), \quad (4.9)$$

where G is a 2×2 matrix propagator for the fields $\phi_+(x)$ and $\phi_-(x)$ that may be obtained from the action (4.5) for the (+) field minus the same action for the (-) field. This propagator cannot be found exactly but it can be obtained perturbatively in powers of the metric perturbations. The unperturbed matrix propagator, G^0 , is the inverse of the kinetic operator $\text{diag}(\square - i\epsilon, -(\square + i\epsilon))$, where we introduced the usual prescriptions for the vacuum state. It has the following components:

$G_{++}^0 = \Delta_F$, $G_{--}^0 = -\Delta_D$, $G_{+-}^0 = -\Delta^+$ and $G_{-+}^0 = \Delta^-$, where Δ_F and Δ_D are the Feynman and Dyson propagators, respectively, and Δ^\pm are the Wightman functions.

Up to second order in the metric perturbations we get,

$$\begin{aligned} S_{\text{IF}}[h_{\mu\nu}^\pm] &\simeq -\frac{i}{2}\text{Tr}(\ln G^0) \\ &+ \frac{i}{2}\text{Tr} \left[V_+^{(1)}G_{++}^0 - V_-^{(1)}G_{--}^0 + V_+^{(2)}G_{++}^0 - V_-^{(2)}G_{--}^0 \right. \\ &\quad \left. - \frac{1}{2}V_+^{(1)}G_{++}^0V_+^{(1)}G_{++}^0 - \frac{1}{2}V_-^{(1)}G_{--}^0V_-^{(1)}G_{--}^0 + V_+^{(1)}G_{+-}^0V_-^{(1)}G_{-+}^0 \right], \end{aligned} \quad (4.10)$$

where we have defined $V_+^{(i)} \equiv V_{++}^{(i)}$ and $V_-^{(i)} \equiv -V_{--}^{(i)}$ ($i = 1, 2$), since the operators $V^{(i)}$ are obviously diagonal. The first trace term is independent of the metric perturbations, its divergences are cancelled by terms which lead to the conformal anomaly. The tadpole terms of type $\text{Tr}(VG)$ involve n -dimensional integrals which are identically zero in dimensional regularization. Thus, all calculation reduces to the computation of the three terms of type $\text{Tr}(VGVG)$. The detailed evaluation of these terms is given in Ref. [68].

After dimensional regularization of the divergent terms and renormalization with the action of the gravitational action (4.8), we obtain the renormalized effective action, (3.3), for the gravitational field. Up to second order it can be written as

$$S_{\text{eff}}[\tilde{g}_{\mu\nu}^\pm] = S_g^R[\tilde{g}_{\mu\nu}^+] - S_g^R[\tilde{g}_{\mu\nu}^-] + S_{\text{IF}}^R[h_{\mu\nu}^\pm], \quad (4.11)$$

where the renormalized influence functional is given by

$$\begin{aligned} S_{\text{IF}}^R[h_{\mu\nu}^\pm] &= -\frac{1}{8} \int d^4x d^4y [h_{\mu\nu}(x)] H^{\mu\nu\alpha\beta}(x, y; \mu) \{h_{\alpha\beta}(y)\} \\ &\quad + \frac{i}{8} \int d^4x d^4y [h_{\mu\nu}(x)] N^{\mu\nu\alpha\beta}(x, y) [h_{\alpha\beta}(y)], \end{aligned} \quad (4.12)$$

and $S_g^R[\tilde{g}_{\mu\nu}^\pm]$ are terms coming from the gravitational action (4.8) after renormalization:

$$\begin{aligned} S_g^R[\tilde{g}_{\mu\nu}] &= \int d^4x \sqrt{-\tilde{g}(x)} \left[\frac{\tilde{R}(x)}{16\pi G_N} - \frac{1}{12} \frac{1}{2880\pi^2} \tilde{R}^2(x) \right] \\ &\quad + \frac{1}{1440\pi^2} \int d^4x \sqrt{-g(x)} \left[G^{\mu\nu}(x) \omega_{;\mu} \omega_{;\nu} + \square_g \omega(\omega_{;\nu} \omega^{;\nu}) + \frac{1}{2} (\omega_{;\mu} \omega^{;\mu})^2 \right] \\ &\quad + \frac{1}{2880\pi^2} \int d^4x \sqrt{-g(x)} \left[R_{\mu\nu\alpha\beta}(x) R^{\mu\nu\alpha\beta}(x) - R_{\mu\nu}(x) R^{\mu\nu}(x) \right] \omega(x). \end{aligned} \quad (4.13)$$

Here we have used the notation of Eq. (3.10) and the kernels $H^{\mu\nu\alpha\beta}$ and $N^{\mu\nu\alpha\beta}$ are given by

$$H^{\mu\nu\alpha\beta}(x, y; \mu) = \frac{2}{3} F_x^{\mu\nu\alpha\beta} H(x - y; \mu), \quad (4.14)$$

where $F_x^{\mu\nu\alpha\beta}$ is the differential operator

$$F_x^{\mu\nu\alpha\beta} \equiv 3F_x^{\mu(\alpha} F_x^{\beta)\nu} - F_x^{\mu\nu} F_x^{\alpha\beta}, \quad F_x^{\mu\nu} \equiv \eta^{\mu\nu} \square_x - \partial_x^\mu \partial_x^\nu, \quad (4.15)$$

and

$$H(x; \mu) = \frac{1}{1920\pi^2} \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot x} \left[\ln \frac{|p^2|}{\mu^2} - i\pi \operatorname{sgn}(p^0) \theta(-p^2) \right]. \quad (4.16)$$

The noise kernel is given by

$$N^{\mu\nu\alpha\beta}(x, y) = \frac{2}{3} F_x^{\mu\nu\alpha\beta} N(x - y), \quad (4.17)$$

where

$$N(x) = \frac{1}{1920\pi} \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot x} \theta(-p^2). \quad (4.18)$$

Note that terms with and without tilde refer to tensors obtained with metrics $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$, respectively.

We are now in a position to derive the Einstein-Langevin equation. Following section 3 we can introduce the stochastic effective action

$$S_{\text{eff}}^s[\tilde{g}_{\mu\nu}^\pm, \xi] = S_g^R[\tilde{g}_{\mu\nu}^+] - S_g^R[\tilde{g}_{\mu\nu}^-] + S_{\text{IF}}^{R,s}[h_{\mu\nu}^\pm, \xi], \quad (4.19)$$

where the renormalized stochastic influence action is given by

$$S_{\text{IF}}^{R,s}[h_{\mu\nu}^\pm, \xi] = \operatorname{Re} S_{\text{IF}}^R[h_{\mu\nu}^\pm] + \frac{1}{2} \int d^4 x \xi^{\mu\nu}(x) [h_{\mu\nu}(x)], \quad (4.20)$$

with the Gaussian stochastic field $\xi^{\mu\nu}$ defined by $\langle \xi^{\mu\nu}(x) \rangle_s = 0$ and

$$\langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(y) \rangle_s = N^{\mu\nu\alpha\beta}(x, y). \quad (4.21)$$

The Einstein-Langevin equation can be obtained by functional derivation according to Eq. (3.15). Note that we have to take the derivative with respect to the physical metric $\tilde{g}_{\mu\nu}$, and we use that for an arbitrary functional $A[\tilde{g}_{\mu\nu}]$,

$$\frac{\delta A[\omega, g_{\mu\nu}]}{\sqrt{-g} \delta g_{\mu\nu}} = e^{\delta\omega} \frac{\delta A[\tilde{g}_{\mu\nu}]}{\sqrt{-\tilde{g}} \delta \tilde{g}_{\mu\nu}}. \quad (4.22)$$

The final result is:

$$\begin{aligned} e^{6\omega} \left[-\frac{1}{8\pi G_N} \left(\tilde{G}_{(0)}^{\mu\nu} + \tilde{G}_{(1)}^{\mu\nu} \right) - \frac{1}{6} \frac{1}{2880\pi^2} \left(\tilde{B}_{(0)}^{\mu\nu} + \tilde{B}_{(1)}^{\mu\nu} \right) \right. \\ \left. + \frac{1}{2880\pi^2} \left(\tilde{H}_{(0)}^{\mu\nu} + \tilde{H}_{(1)}^{\mu\nu} \right) - \frac{1}{1440\pi^2} \tilde{R}_{\alpha\beta}^{(0)} \tilde{C}^{\mu\alpha\nu\beta}_{(1)} \right] \\ - \frac{3}{720\pi^2} (C_{(1)}^{\mu\alpha\nu\beta} \omega)_{,\alpha\beta} - \int d^4 y A_{(1)}^{\mu\nu}(y) H(x - y; \mu) + \xi^{\mu\nu} = O(h_{\mu\nu}^2), \end{aligned} \quad (4.23)$$

where the (0) and (1) sub-indices refer to the zero and first orders terms, respectively, in the metric perturbation $h_{\mu\nu}$. The tensors $A^{\mu\nu}(x)$ and $B^{\mu\nu}(x)$ are given by Eqs. (2.7) and (2.8), respectively, and $H^{\mu\nu}(x)$ is given by

$$H^{\mu\nu} = -R^{\mu\alpha} R_{\alpha}{}^{\nu} + \frac{2}{3} R R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{4} g^{\mu\nu} R^2. \quad (4.24)$$

When comparing the Einstein-Langevin Eq. (4.23) with Eq. (2.12) we recall that in our renormalization scheme [see Eq. (4.8)], we have implicitly fixed the arbitrary parameter β , whereas the

parameter α appears related to the parameter μ . In fact, if we change μ by μ' in the kernel (4.16) we have

$$H(x-y; \mu) = H(x-y; \mu') + \frac{1}{1920\pi^2} \delta^{(4)}(x-y) \ln \frac{\mu^2}{\mu'^2}, \quad (4.25)$$

therefore the arbitrariness of α corresponds to that of the renormalization parameter μ . To be specific, one must assume that $\alpha(\mu)$ and that if we change μ by μ' also α changes appropriately so that the physical parameters in the Einstein-Langevin equation do not change. Note that $\tilde{A}_{(0)}^{\mu\nu} = 0$ because $\tilde{A}^{\mu\nu}$ is obtained by taking the functional derivative of an action density proportional to the square of the Weyl tensor, see (2.7), but the conformal symmetry is broken only by the metric perturbations $h_{\mu\nu}$, at zero order in the perturbation the physical metric is conformally flat and the Weyl tensor is zero. Since the parameters α and β as well as the gravitational constant can only be fixed by experiment we will introduce a (rescaled) parameter β in our final equations.

Let us write the Einstein-Langevin Eq. (4.23) with the effective source term on the right-hand side as

$$\tilde{G}^{\mu\nu} = 8\pi G_N \left(\langle T^{\mu\nu} \rangle + e^{-6\omega} \xi^{\mu\nu} \right), \quad (4.26)$$

where $\langle T^{\mu\nu} \rangle$ is the vacuum expectation value of the stress-energy tensor of the quantum field up to first order in $h_{\mu\nu}$. It is given by

$$\begin{aligned} \langle T_{(0)}^{\mu\nu} \rangle &= \frac{1}{2880\pi^2} \left[\tilde{H}_{(0)}^{\mu\nu} - \frac{\beta}{6} \tilde{B}_{(0)}^{\mu\nu} \right] \\ \langle T_{(1)}^{\mu\nu} \rangle &= \frac{1}{2880\pi^2} \left[\tilde{H}_{(1)}^{\mu\nu} - \frac{\beta}{6} \tilde{B}_{(1)}^{\mu\nu} - 2\tilde{R}_{\alpha\beta}^{(0)} \tilde{C}^{\mu\alpha\nu\beta}_{(1)} \right. \\ &\quad \left. - 12e^{-6\omega} (C_{(1)}^{\mu\alpha\nu\beta} \omega)_{,\alpha\beta} \right] - e^{-6\omega} \int d^4y A_{(1)}^{\mu\nu}(y) H(x-y; \mu), \end{aligned} \quad (4.27)$$

where the rescaled β parameter has been introduced. This tensor was first derived in Ref. [68] using the CTP formalism, but it had been derived by other means in Refs. [55]. To summarize, the stochastic equation (4.26) is the semiclassical Einstein-Langevin equation for weakly inhomogeneous perturbations on spatially-flat FRW spacetimes with a conformally coupled massless scalar field. Here, following Ref. [41] we have derived a stochastic correction to the vacuum expectation value of the quantum field stress-energy tensor (4.27), which accounts for the noise associated with the fluctuations of the stress-energy tensor on the homogeneous background spacetime.

We can recover the trace anomaly result [5]. In fact, to first order in $h_{\mu\nu}$ we have:

$$\begin{aligned} \langle T^\mu{}_\mu \rangle &= \langle T_{(0)\mu}^\mu \rangle + \langle T_{(1)\mu}^\mu \rangle + O(h_{\mu\nu}^2) \\ &= \frac{1}{2880\pi^2} \left[\beta \square_{\tilde{g}} \tilde{R} + \left(\tilde{R}^{\mu\nu} \tilde{R}_{\mu\nu} - \frac{1}{3} \tilde{R}^2 \right) \right] + O(h_{\mu\nu}^2). \end{aligned} \quad (4.28)$$

Note that due to the conformal invariance of $\int d^4x \sqrt{-g} \tilde{C}_{\mu\nu\alpha\beta} \tilde{C}^{\mu\nu\alpha\beta}$ the tensor $\tilde{A}^{\mu\nu}$ is traceless.

As we have seen in Sec. 2 the stochastic correction to the stress-energy tensor is traceless for conformal fields and has vanishing divergence to first order in the metric perturbations. These properties can be easily checked in this case. That $\langle \xi^\mu \rangle_s = 0$ follows directly from Eq. (4.21) and (4.17) by noticing that $\langle N_{\mu}^{\mu\alpha\beta}(x, y) \rangle = 0$ as a consequence of $F^{\mu(\alpha} F^{\beta)}_{\mu} = \square F^{\alpha\beta}$ and $F^\mu{}_\mu = 3\square$. Furthermore, using $\tilde{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}$ and that $\xi^{\mu\nu}$ is symmetric and traceless, it is easy to see that $\tilde{\nabla}_\nu (e^{-6\omega} \xi^{\mu\nu}) = e^{-6\omega} \nabla_\nu \xi^{\mu\nu}$. Then, from Eq. (4.21) and the symmetries of ξ , we obtain that

$\nabla_\nu \xi^{\mu\nu} = O(h_{\mu\nu})$. It is thus consistent to write this term on the right-hand side of Einstein equations and consider it as a correction of order higher than $\langle T_{(0)}^{\mu\nu} \rangle$ (note that $\tilde{\nabla}_\nu \langle T_{(0)}^{\mu\nu} \rangle = O(h_{\mu\nu}^2)$).

Taking the mean value of the Einstein-Langevin equation with respect to the tensor field source $\xi^{\mu\nu}$ with Gaussian probability distributions (4.21), we obtain the semiclassical Einstein equation. This equation can be used to study the stability of the zero order semiclassical Eq. (4.26):

$$\tilde{G}_{(0)}^{\mu\nu} = 8\pi G_N \langle T_{(0)}^{\mu\nu} \rangle. \quad (4.29)$$

If we take the value $\beta = 1$, which is the value obtained in our renormalization scheme, this stress-energy tensor $\langle T_{(0)}^{\mu\nu} \rangle$ for a scalar field in a conformally flat spacetime exactly agrees with that found by other techniques [5]. But in general, as remarked earlier, the coefficient β is arbitrary and should be determined by experiment. On the other hand since the tensor $\tilde{A}_{(0)}^{\mu\nu} = 0$ there is no α coefficient [see Eq. (2.12)]. Moreover, the coefficient of the tensor $\tilde{H}_{(0)}^{\mu\nu}$ is fixed, its value differs for different types and number of fields, it has been computed here for a scalar field. Note that the tensor $\tilde{H}^{\mu\nu}$ is conserved only in conformally flat spacetimes and cannot be obtained by functional derivation of a geometrical term in the action, so that it plays a very different role than the tensors $\tilde{A}^{\mu\nu}$ and $\tilde{B}^{\mu\nu}$. Equation (4.29) can be solved to find the conformal factor $\omega(\eta)$, it was shown by Starobinsky [88, 89] that this equation describes the so-called trace anomaly driven inflation. Note that we can use the Einstein-Langevin equation (4.26) to compute the two-point correlations for the metric perturbations induced by the stress-energy tensor fluctuations. These correlations are relevant for the generation of primordial inhomogeneities in this inflationary scenario [90]. This problem has been treated recently from the stochastic gravity perspective in Ref. [30] for inflationary models driven by scalar fields.

It is interesting to notice also that the imaginary term in the regularized influence action (4.12) can be written after an integration by parts in the following alternative form [41]

$$\text{Im } S_{IF}^R[h_{\mu\nu}^\pm] = \frac{1}{2} \int d^4x d^4y [C_{\mu\nu\alpha\beta}(x)] N(x - y) [C^{\mu\nu\alpha\beta}(y)], \quad (4.30)$$

which shows that the noise couples to the conformal tensor. Here we have used the notation (3.10) for the square brackets. The fact that the stochastic source couples to the conformal tensor is not a surprise. For a conformal quantum field non trivial quantum effects are a consequence of breaking the conformal symmetry of the spacetime, which is characterized by the conformal tensor. For instance, it is known that the probability density of pair creation in this case [91], or in the presence of small anisotropy [40], is determined by the square of the Weyl tensor. Thus, as it has been shown in Refs. [38, 27] there is a direct relation between particle creation and noise.

5 Black Hole Backreaction Problem

The celebrated Hawking effect of particle creation from black holes is constructed from a quantum field theory in curved spacetime (QFTCST) framework. The oft-mentioned ‘black hole evaporation’ referring to the reduction of the mass of a black hole due to particle creation must entail backreaction considerations. Backreaction of Hawking radiation [81, 82, 83, 92, 93, 94] could alter the evolution of the background spacetime and change the nature of its end state, more drastically so for Planck size black holes. Because of the higher symmetry in cosmological spacetimes, backreaction studies of processes therein have progressed further than the corresponding black hole problems, which to

a large degree is still concerned with finding the right approximations for the regularized energy momentum tensor [95, 96, 97, 98, 99]⁵ for even the simplest spacetimes such as the spherically symmetric family including the important Schwarzschild metric. Though arduous and demanding, the effort continues on because of the importance of backreaction effects of Hawking radiation in black holes. They are expected to address some of the most basic issues such as black hole thermodynamics [11, 109, 110, 111] and the black hole end-state and information loss puzzles [112].

Here we wish to address the black hole backreaction problem with new insights provided by stochastic semiclassical gravity (SSG). (For the latest developments see reviews, e.g., [114, 1, 2, 3]). It is not our intention to seek better approximations for the regularized energy momentum tensor, but to point out new ingredients lacking in the existing framework based on semiclassical gravity (SCG). In particular one needs to consider both the dissipation and the fluctuations aspects in the back reaction of particle creation or vacuum polarization.

In a short note [115] Raval, Sinha and one of us (HRS) discussed the formulation of the problem in this new light, commented on some shortcomings of existing works, and sketched the strategy [116] behind our own approach to the black hole fluctuations and backreaction problem. Here we focus only on the class of quasi-static black holes, leaving the more demanding dynamical collapse problem to a later exposition. Thus we only address the first set of major issues mentioned above.

From the new perspective provided by statistical field theory and stochastic gravity, it is not difficult to postulate that backreaction effect is the manifestation of a fluctuation-dissipation relation (FDR) [117]. This was first conjectured by Candelas and Sciamia [118] for a dynamic Kerr black hole emitting Hawking radiation, and Mottola [119] for a static black hole (in a box) in quasi-equilibrium with its radiation via linear response theory (LRT) [120]. While the FDR in a LRT captures the response of the system (e.g., dissipation of the black hole) to the environment (in these cases the matter field) linear response theory (in the way it is commonly presented in statistical thermodynamics) cannot provide a full description of self-consistent backreaction on at least two counts: First, because it is usually based on the assumption of a specified background spacetime (static in this case) and state (thermal) of the matter field(s) (e.g., [119]). The spacetime and the state of matter should be determined in a self-consistent manner by their dynamics and mutual influence. Second, the fluctuation part represented by the noise kernel is amiss (e.g., [35]) This is also a problem in the FDR proposed by Candelas and Sciamia [118] (see below). As will be shown in an explicit example later, the back reaction is intrinsically a dynamic process. The Einstein-Langevin equation in stochastic gravity overcomes both of these deficiencies.

For Candelas and Sciamia [118], the classical formula they showed relating the dissipation in area linearly to the squared absolute value of the shear amplitude is suggestive of a fluctuation-dissipation relation. When the gravitational perturbations are quantized (they choose the quantum

⁵The latest important work is that of Hiscock, Larson and Anderson [99] on backreaction in the interior of a black hole, where one can find a concise summary of earlier work. To name a few of the important landmarks in this endeavor (this is adopted from [99]), Howard and Candelas [100, 101] have computed the stress-energy of a conformally invariant scalar field in the Schwarzschild geometry. Jensen and Ottewill [102] have computed the vacuum stress-energy of a massless vector field in Schwarzschild. Approximation methods have been developed by Page, Brown, and Ottewill [103, 104, 105] for conformally invariant fields in Schwarzschild spacetime, Frolov and Zel'nikov [106] for conformally invariant fields in a general static spacetime, Anderson, Hiscock and Samuel [97] for massless arbitrarily coupled scalar fields in a general static spherically symmetric spacetime. Furthermore the DeWitt-Schwinger approximation has been derived by Frolov and Zel'nikov [107, 108] for massive fields in Kerr spacetime, Anderson Hiscock and Samuel [97] for a general (arbitrary curvature coupling and mass) scalar field in a general static spherically symmetric spacetime and have applied their method to the Reissner-Nordström geometry [98].

state to be the Unruh vacuum) they argue that it approximates a flux of radiation from the hole at large radii. Thus the dissipation in area due to the Hawking flux of gravitational radiation is allegedly related to the quantum fluctuations of gravitons. HRS’s criticism [115] is that their’s is not an FDR in the truly statistical mechanical sense because it does not relate dissipation of a certain quantity (in this case, horizon area) to the fluctuations of *the same quantity*. To do so would require one to compute the two point function of the area, which, being a four-point function of the graviton field, is related to a two-point function of the stress tensor. The stress tensor is the true “generalized force” acting on the spacetime via the equations of motion, and the dissipation in the metric must eventually be related to the fluctuations of this generalized force for the relation to qualify as an FDR.

From this reasoning, we see that the stress energy bi-tensor and its vacuum expectation value known as the noise kernel, are the new ingredients in backreaction considerations. But these are exactly the centerpiece in stochastic gravity. Therefore the correct framework to address semiclassical backreaction problems is stochastic gravity theory, where fluctuations and dissipation are the equally essential components. The noise kernel for quantum fields in Minkowski and de Sitter spacetime has been carried out by Martin, Roura and Verdaguer [27, 28, 30], for thermal fields in black hole spacetime and scalar fields in general spacetimes by Campos, Hu and Phillips [72, 24, 25, 26]. Earlier, for cosmological backreaction problems Hu and Sinha [40] derived a generalized expression relating dissipation (of anisotropy in a Bianchi Type I universes) and fluctuations (measured by particle numbers created in neighboring histories). This example shows that one can understand the backreaction of particle creation as a manifestation of a (generalized) FDR.

As an illustration of the application of stochastic gravity theory we outline the steps in a black hole backreaction calculation, focusing on the manageable quasi-static class. We adopt the Hartle-Hawking picture [122] where the black hole is bathed eternally – actually in quasi-thermal equilibrium – in the Hawking radiance it emits. It is described here by a massless scalar quantum field at the Hawking temperature. As is well-known, this quasi-equilibrium condition is possible only if the black hole is enclosed in a box of size suitably larger than the event horizon. We can divide our consideration into the far field case and the near horizon case. Campos and Hu [72] have treated a relativistic thermal plasma in a weak gravitational field. Since the far field limit of a Schwarzschild metric is just the perturbed Minkowski spacetime, one can perform a perturbation expansion off hot flat space using the thermal Green functions [123] ⁶. Hot flat space has been studied before for various purposes. See e.g., [124, 125, 126]. Campos and Hu derived a stochastic CTP effective action and from it an equation of motion, the Einstein Langevin equation, for the dynamical effect of a scalar quantum field on a background spacetime ⁷. Recently Sinha, Raval and Hu [116] outlined a strategy for treating the near horizon case, following the same scheme of Campos and Hu. In both cases two new terms appear which are absent in SCG considerations: a nonlocal dissipation and a (generally colored) noise kernel. When one takes the noise average one recovers York’s [83] semiclassical equations for radially perturbed quasi-static black holes. For the near horizon case one cannot obtain the full details yet, because the Green function for a scalar

⁶Strictly speaking the location of the box holding the black hole in equilibrium with its thermal radiation is as far as one can go, thus the metric may not reach the perturbed Minkowski form. But one can also put the black hole and its radiation in an anti-de Sitter space [113], which contains such a region.

⁷To perform calculations leading to the Einstein-Langevin equation one needs to begin with a self-consistent solution of the semiclassical Einstein equation for the thermal field and the perturbed background spacetime. For a black hole background, a semiclassical gravity solution is provided by York [83]. For a Robertson-Walker background with thermal fields it is given by Hu [127].

field in the Schwarzschild metric comes only in an approximate form (e.g. Page approximation [103]), which, though reasonably accurate for the stress tensor, fails poorly for the noise kernel [25, 26]. In addition a formula is derived in [116] expressing the CTP effective action in terms of the Bogolyubov coefficients. Since it measures not only the number of particles created, but also the difference of particle creation in alternative histories, this provides a useful avenue to explore the wider set of issues in black hole physics related to noise and fluctuations.

Since backreaction calculations in semiclassical gravity has been under study for a much longer time than in stochastic gravity we will concentrate on explaining how the new stochastic features arise from the framework of semiclassical gravity, i.e., noise and fluctuations and their consequences. Technically the goal is to obtain an influence action for this model of a black hole coupled to a scalar field and to derive an Einstein-Langevin equation from it. As a by-product, from the fluctuation-dissipation relation, one can derive the vacuum susceptibility function and the isothermal compressibility function for black holes, two quantities of fundamental interest in characterizing the nonequilibrium thermodynamic properties of black holes.

5.1 The Model

In this model the black hole spacetime is described by a spherically symmetric static metric with line element of the following general form written in advanced time Eddington-Finkelstein coordinates

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\psi} \left(1 - \frac{2m}{r}\right) dv^2 + 2e^{2\psi} dvdr + r^2 d\Omega^2 \quad (5.1)$$

where $\psi = \psi(r)$ and $m = m(r)$, $v = t + r + 2M \ln(\frac{r}{2M} - 1)$ and $d\Omega^2$ is the line element on the two sphere. Hawking radiation is described by a massless, conformally coupled quantum scalar field ϕ with the classical action

$$S_m[\phi, g_{\mu\nu}] = -\frac{1}{2} \int d^n x \sqrt{-g} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi(n) R \phi^2] \quad (5.2)$$

where $\xi(n) = \frac{(n-2)}{4(n-1)}$ (n is the dimension of spacetime) and R is the curvature scalar of the spacetime it lives in.

Let us consider linear perturbations $h_{\mu\nu}$ off a background Schwarzschild metric $g_{\mu\nu}^{(0)}$

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} \quad (5.3)$$

with standard line element

$$(ds^2)^0 = \left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \quad (5.4)$$

We look for this class of perturbed metrics in the form given by (5.1), (thus restricting our consideration only to spherically symmetric perturbations):

$$e^\psi \simeq 1 + \epsilon \rho(r) \quad (5.5)$$

and

$$m \simeq M[1 + \epsilon \mu(r)] \quad (5.6)$$

where $\frac{\epsilon}{\lambda M^2} = \frac{1}{3}aT_H^4$; $a = \frac{\pi^2}{30}$; $\lambda = 90(8^4)\pi^2$. T_H is the Hawking temperature. This particular parametrization of the perturbation is chosen following York's [83] notation. Thus the only non-zero components of $h_{\mu\nu}$ are

$$h_{vv} = - \left(\left(1 - \frac{2M}{r}\right) 2\epsilon\rho(r) + \frac{2M\epsilon\mu(r)}{r} \right) \quad (5.7)$$

and

$$h_{vr} = \epsilon\rho(r) \quad (5.8)$$

So this represents a metric with small static and radial perturbations about a Schwarzschild black hole. The initial quantum state of the scalar field is taken to be the Hartle Hawking vacuum, which is essentially a thermal state at the Hawking temperature and it represents a black hole in (unstable) thermal equilibrium with its own Hawking radiation. In the far field limit, the gravitational field is described by a linear perturbation from Minkowski spacetime. In equilibrium the thermal bath can be characterized by a relativistic fluid with a four-velocity (time-like normalized vector field) u^μ , and temperature in its own rest frame β^{-1} .

To facilitate later comparisons with our program we briefly recall York's work [83].⁸ He considered the semiclassical Einstein equation

$$G_{\mu\nu}(g_{\alpha\beta}) = \kappa\langle T_{\mu\nu} \rangle \quad (5.9)$$

with $G_{\mu\nu} \simeq G_{\mu\nu}^{(0)} + \delta G_{\mu\nu}$ where $G_{\mu\nu}^{(0)}$ is the Einstein tensor for the background spacetime. The zeroth order solution gives a background metric in empty space, i.e, the Schwarzschild metric. $\delta G_{\mu\nu}$ is the linear correction to the Einstein tensor in the perturbed metric. The semiclassical Einstein equation in this approximation therefore reduces to

$$\delta G_{\mu\nu}(g^{(0)}, h) = \kappa\langle T_{\mu\nu} \rangle \quad (5.10)$$

York solved this equation to first order by using the expectation value of the energy momentum tensor for a conformally coupled scalar field in the Hartle-Hawking vacuum in the unperturbed (Schwarzschild) spacetime on the right hand side and using (5.7) and (5.8) to calculate $\delta G_{\mu\nu}$ on the left hand side. Unfortunately, no exact analytical expression is available for the $\langle T_{\mu\nu} \rangle$ in a Schwarzschild metric with the quantum field in the Hartle-Hawking vacuum that goes on the right hand side. York therefore uses the approximate expression given by Page [103] which is known to give excellent agreement with numerical results. Page's approximate expression for $\langle T_{\mu\nu} \rangle$ was constructed using a thermal Feynman Green's function obtained by a conformal transformation of a WKB approximated Green's function for an optical Schwarzschild metric. York then solves the semiclassical Einstein equation (5.10) self consistently to obtain the corrections to the background metric induced by the backreaction encoded in the functions $\mu(r)$ and $\rho(r)$. There was no mention of fluctuations or its effects. As we shall see, in the language of the previous section, the semiclassical gravity procedure which York followed working at the equation of motion level is equivalent to looking at the noise-averaged backreaction effects.

⁸See also work by Hochberg and Kephart [92] for a massless vector field, Hochberg, Kephart and York [93] for a massless spinor field, and Anderson, Hiscock, Whitesell, and York [94] for a quantized massless scalar field with arbitrary coupling to spacetime curvature

5.2 CTP Effective Action for the Black Hole

We first derive the CTP effective action for the model described in the previous section. Using the metric (5.4) (and neglecting the surface terms that appear in an integration by parts) we have the action for the scalar field written perturbatively as

$$S_m[\phi, h_{\mu\nu}] = \frac{1}{2} \int d^n x \sqrt{-g^{(0)}} \phi \left[\square^{(0)} + V^{(1)} + V^{(2)} + \dots \right] \phi, \quad (5.11)$$

where the first and second order perturbative operators $V^{(1)}$ and $V^{(2)}$ are given by

$$\begin{aligned} V^{(1)} &\equiv -\frac{1}{\sqrt{-g^{(0)}}} \left\{ [\partial_\mu (\sqrt{-g^{(0)}} \bar{h}^{\mu\nu}(x))] \partial_\nu + \bar{h}^{\mu\nu}(x) \partial_\mu \partial_\nu + \xi(n) R^{(1)}(x) \right\}, \\ V^{(2)} &\equiv -\frac{1}{\sqrt{-g^{(0)}}} \left\{ [\partial_\mu (\sqrt{-g^{(0)}} \hat{h}^{\mu\nu}(x))] \partial_\nu + \hat{h}^{\mu\nu}(x) \partial_\mu \partial_\nu - \xi(n) (R^{(2)}(x) + \frac{1}{2} h(x) R^{(1)}(x)) \right\} \end{aligned} \quad (5.12)$$

In the above expressions, $R^{(k)}$ is the k -order term in the perturbation $h_{\mu\nu}(x)$ of the scalar curvature R and $\bar{h}_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ denote a linear and a quadratic combination of the perturbation, respectively,

$$\begin{aligned} \bar{h}_{\mu\nu} &\equiv h_{\mu\nu} - \frac{1}{2} h g_{\mu\nu}^{(0)}, \\ \hat{h}_{\mu\nu} &\equiv h_\mu^\alpha h_{\alpha\nu} - \frac{1}{2} h h_{\mu\nu} + \frac{1}{8} h^2 g_{\mu\nu}^{(0)} - \frac{1}{4} h_{\alpha\beta} h^{\alpha\beta} g_{\mu\nu}^{(0)}. \end{aligned} \quad (5.13)$$

From quantum field theory in curved spacetime considerations discussed above we take the following action for the gravitational field:

$$\begin{aligned} S_g[g_{\mu\nu}] &= \frac{1}{(16\pi G)^{\frac{n-2}{2}}} \int d^n x \sqrt{-g(x)} R(x) + \frac{\alpha \bar{\mu}^{n-4}}{4(n-4)} \int d^n x \sqrt{-g(x)} \\ &\quad \left\{ 3R_{\mu\nu\alpha\beta}(x) R^{\mu\nu\alpha\beta}(x) - \left[1 - 360 \left(\xi(n) - \frac{1}{6} \right)^2 \right] R^2(x) \right\}. \end{aligned} \quad (5.14)$$

The first term is the classical Einstein-Hilbert action and the second term is the counterterm in four dimensions used to renormalize the divergent effective action. In this action $\ell_P^2 = 16\pi G_N$, $\alpha = (2880\pi^2)^{-1}$ and $\bar{\mu}$ is an arbitrary mass scale.

We are interested in computing the CTP effective action (5.11) for the matter action and when the field ϕ is initially in the Hartle-Hawking vacuum. This is equivalent to saying that the initial state of the field is described by a thermal density matrix at a finite temperature $T = T_H$. The CTP effective action at finite temperature $T \equiv 1/\beta$ for this model is given by (for details see [72])

$$S_{rmeff}^\beta[h_{\mu\nu}^\pm] = S_g[h_{\mu\nu}^+] - S_g[h_{\mu\nu}^-] - \frac{i}{2} \text{Tr} \{ \ln \bar{G}_{ab}^\beta[h_{\mu\nu}^\pm] \}, \quad (5.15)$$

where \pm denote the forward and backward time path of the CTP formalism and $\bar{G}_{ab}^\beta[h_{\mu\nu}^\pm]$ is the complete 2×2 matrix propagator (a and b take \pm values: G_{++}, G_{+-} and G_{--} correspond to the Feynman, Wightman and Schwinger Greens functions respectively) with thermal boundary conditions for the differential operator $\sqrt{-g^{(0)}}(\square + V^{(1)} + V^{(2)} + \dots)$. The actual form of \bar{G}_{ab}^β

cannot be explicitly given. However, it is easy to obtain a perturbative expansion in terms of $V_{ab}^{(k)}$, the k -order matrix version of the complete differential operator defined by $V_{\pm\pm}^{(k)} \equiv \pm V_{\pm}^{(k)}$ and $V_{\pm\mp}^{(k)} \equiv 0$, and G_{ab}^{β} , the thermal matrix propagator for a massless scalar field in Schwarzschild spacetime. To second order \bar{G}_{ab}^{β} reads,

$$\bar{G}_{ab}^{\beta} = G_{ab}^{\beta} - G_{ac}^{\beta} V_{cd}^{(1)} G_{db}^{\beta} - G_{ac}^{\beta} V_{cd}^{(2)} G_{db}^{\beta} + G_{ac}^{\beta} V_{cd}^{(1)} G_{de}^{\beta} V_{ef}^{(1)} G_{fb}^{\beta} + \dots \quad (5.16)$$

Expanding the logarithm and dropping one term independent of the perturbation $h_{\mu\nu}^{\pm}(x)$, the CTP effective action may be perturbatively written as,

$$\begin{aligned} S_{\text{eff}}^{\beta}[h_{\mu\nu}^{\pm}] &= S_g[h_{\mu\nu}^{+}] - S_g[h_{\mu\nu}^{-}] \\ &+ \frac{i}{2} \text{Tr}[V_{+}^{(1)} G_{++}^{\beta} - V_{-}^{(1)} G_{--}^{\beta} + V_{+}^{(2)} G_{++}^{\beta} - V_{-}^{(2)} G_{--}^{\beta}] \\ &- \frac{i}{4} \text{Tr}[V_{+}^{(1)} G_{++}^{\beta} V_{+}^{(1)} G_{++}^{\beta} + V_{-}^{(1)} G_{--}^{\beta} V_{-}^{(1)} G_{--}^{\beta} - 2V_{+}^{(1)} G_{+-}^{\beta} V_{-}^{(1)} G_{-+}^{\beta}]. \end{aligned} \quad (5.17)$$

In computing the traces, some terms containing divergences are canceled using counterterms introduced in the classical gravitational action after dimensional regularization.

5.3 Near Flat Case

At this point we divide our considerations into two cases. In the far field limit $h_{\mu\nu}$ represent perturbations about flat space, i.e., $g_{\mu\nu}^{(0)} = \eta_{\mu\nu}$. The exact “unperturbed” thermal propagators for scalar fields are known, i.e., the Euclidean propagator with periodicity β . Using the Fourier transformed form (those quantities are denoted with a tilde) of the thermal propagators $\tilde{G}_{ab}^{\beta}(k)$, the trace terms of the form $\text{Tr}[V_a^{(1)} G_{mn}^{\beta} V_b^{(1)} G_{rs}^{\beta}]$ can be written as [72],

$$\text{Tr}[V_a^{(1)} G_{mn}^{\beta} V_b^{(1)} G_{rs}^{\beta}] = \int d^n x d^n x' h_{\mu\nu}^a(x) h_{\alpha\beta}^b(x') \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} e^{ik \cdot (x-x')} \tilde{G}_{mn}^{\beta}(k+q) \tilde{G}_{rs}^{\beta}(q) T^{\mu\nu, \alpha\beta}(q, k), \quad (5.18)$$

where the tensor $T^{\mu\nu, \alpha\beta}(q, k)$ is defined in [72] after an expansion in terms of a basis of 14 tensors [125]. In particular, the last trace of (5.17) may be split in two different kernels $N^{\mu\nu, \alpha\beta}(x - x')$ and $D^{\mu\nu, \alpha\beta}(x - x')$,

$$\frac{i}{2} \text{Tr}[V_{+}^{(1)} G_{+-}^{\beta} V_{-}^{(1)} G_{-+}^{\beta}] = - \int d^4 x d^4 x' h_{\mu\nu}^{+}(x) h_{\alpha\beta}^{-}(x') [D^{\mu\nu, \alpha\beta}(x - x') + iN^{\mu\nu, \alpha\beta}(x - x')]. \quad (5.19)$$

One can express the Fourier transforms of these kernels, respectively, as

$$\begin{aligned} \tilde{N}^{\mu\nu, \alpha\beta}(k) &= \pi^2 \int \frac{d^4 q}{(2\pi)^4} \{ \theta(k^o + q^o) \theta(-q^o) + \theta(-k^o - q^o) \theta(q^o) + n_{\beta}(|q^o|) + n_{\beta}(|k^o + q^o|) \\ &+ 2n_{\beta}(|q^o|) n_{\beta}(|k^o + q^o|) \} \delta(q^2) \delta[(k+q)^2] T^{\mu\nu, \alpha\beta}(q, k), \end{aligned} \quad (5.20)$$

$$\begin{aligned} \tilde{D}^{\mu\nu, \alpha\beta}(k) &= -i\pi^2 \int \frac{d^4 q}{(2\pi)^4} \{ \theta(k^o + q^o) \theta(-q^o) - \theta(-k^o - q^o) \theta(q^o) + sg(k^o + q^o) n_{\beta}(|q^o|) \\ &- sg(q^o) n_{\beta}(|k^o + q^o|) \} \delta(q^2) \delta[(k+q)^2] T^{\mu\nu, \alpha\beta}(q, k). \end{aligned} \quad (5.21)$$

Using the property $T^{\mu\nu,\alpha\beta}(q, k) = T^{\mu\nu,\alpha\beta}(-q, -k)$, it is easy to see that $N^{\mu\nu,\alpha\beta}(x - x')$ is symmetric and $D^{\mu\nu,\alpha\beta}(x - x')$ antisymmetric in their arguments; that is, $N^{\mu\nu,\alpha\beta}(x) = N^{\mu\nu,\alpha\beta}(-x)$ and $D^{\mu\nu,\alpha\beta}(x) = -D^{\mu\nu,\alpha\beta}(-x)$.

The physical meanings of these kernels can be extracted if we write the renormalized CTP effective action at finite temperature (5.17) in an influence functional form [87]. N , the imaginary part of the CTP effective action can be identified with the noise kernel and D , the antisymmetric piece of the real part, with the dissipation kernel. Campos and Hu [72] have shown that these kernels identified as such indeed satisfy a thermal FDR.

If we denote the difference and the sum of the perturbations $h_{\mu\nu}^{\pm}$ defined along each branch C_{\pm} of the complex time path of integration C by $[h_{\mu\nu}] \equiv h_{\mu\nu}^+ - h_{\mu\nu}^-$ and $\{h_{\mu\nu}\} \equiv h_{\mu\nu}^+ + h_{\mu\nu}^-$, respectively, the influence functional form of the thermal CTP effective action may be written to second order in $h_{\mu\nu}$ as,

$$\begin{aligned} S_{\text{eff}}^{\beta}[h_{\mu\nu}^{\pm}] \simeq & \frac{1}{2(16\pi G_N)} \int d^4x d^4x' [h_{\mu\nu}](x) L_{(o)}^{\mu\nu,\alpha\beta}(x - x') \{h_{\alpha\beta}\}(x') \\ & + \frac{1}{2} \int d^4x [h_{\mu\nu}](x) T_{(\beta)}^{\mu\nu} \\ & + \frac{1}{2} \int d^4x d^4x' [h_{\mu\nu}](x) H^{\mu\nu,\alpha\beta}(x - x') \{h_{\alpha\beta}\}(x') \\ & - \frac{1}{2} \int d^4x d^4x' [h_{\mu\nu}](x) D^{\mu\nu,\alpha\beta}(x - x') \{h_{\alpha\beta}\}(x') \\ & + \frac{i}{2} \int d^4x d^4x' [h_{\mu\nu}](x) N^{\mu\nu,\alpha\beta}(x - x') [h_{\alpha\beta}](x'). \end{aligned} \quad (5.22)$$

The first line is the Einstein-Hilbert action to second order in the perturbation $h_{\mu\nu}^{\pm}(x)$. $L_{(o)}^{\mu\nu,\alpha\beta}(x)$ is a symmetric kernel (i.e. $L_{(o)}^{\mu\nu,\alpha\beta}(x) = L_{(o)}^{\mu\nu,\alpha\beta}(-x)$). In the near flat case its Fourier transform is given by

$$\tilde{L}_{(o)}^{\mu\nu,\alpha\beta}(k) = \frac{1}{4} \left[-k^2 T_1^{\mu\nu,\alpha\beta}(q, k) + 2k^2 T_4^{\mu\nu,\alpha\beta}(q, k) + T_8^{\mu\nu,\alpha\beta}(q, k) - 2T_{13}^{\mu\nu,\alpha\beta}(q, k) \right]. \quad (5.23)$$

The fourteen elements of the tensor basis $T_i^{\mu\nu,\alpha\beta}(q, k)$ ($i = 1, \dots, 14$) are defined in [125]. The second is a local term linear in $h_{\mu\nu}^{\pm}(x)$. Only when far away from the hole that it takes the form of the stress tensor of massless scalar particles at temperature β^{-1} , which has the form of a perfect fluid stress-energy tensor

$$T_{(\beta)}^{\mu\nu} = \frac{\pi^2}{30\beta^4} \left[u^{\mu} u^{\nu} + \frac{1}{3} (\eta^{\mu\nu} + u^{\mu} u^{\nu}) \right], \quad (5.24)$$

where u^{μ} is the four-velocity of the plasma ⁹ and the factor $\frac{\pi^2}{30\beta^4}$ is the familiar thermal energy density for massless scalar particles at temperature β^{-1} . In the third line, the Fourier transform of the symmetric kernel $H^{\mu\nu,\alpha\beta}(x)$ can be expressed as

$$\tilde{H}^{\mu\nu,\alpha\beta}(k) = -\frac{\alpha k^4}{4} \left\{ \frac{1}{2} \ln \frac{|k^2|}{\mu^2} Q^{\mu\nu,\alpha\beta}(k) + \frac{1}{3} \bar{Q}^{\mu\nu,\alpha\beta}(k) \right\}$$

⁹In the far field limit, taking into account the four-velocity u^{μ} of the fluid, a manifestly Lorentz-covariant approach to thermal field theory may be used [128]. However, in order to simplify the involved tensorial structure we work in the co-moving coordinate system of the fluid where $u^{\mu} = (1, 0, 0, 0)$.

$$\begin{aligned}
& + \frac{\pi^2}{180\beta^4} \left\{ -T_1^{\mu\nu,\alpha\beta}(u, k) - 2T_2^{\mu\nu,\alpha\beta}(u, k) + T_4^{\mu\nu,\alpha\beta}(u, k) + 2T_5^{\mu\nu,\alpha\beta}(u, k) \right\} \\
& + \frac{\xi}{96\beta^2} \left\{ k^2 T_1^{\mu\nu,\alpha\beta}(u, k) - 2k^2 T_4^{\mu\nu,\alpha\beta}(u, k) - T_8^{\mu\nu,\alpha\beta}(u, k) + 2T_{13}^{\mu\nu,\alpha\beta}(u, k) \right\} \\
& + \pi \int \frac{d^4 q}{(2\pi)^4} \left\{ \delta(q^2) n_\beta(|q^o|) \mathcal{P} \left[\frac{1}{(k+q)^2} \right] + \delta[(k+q)^2] n_\beta(|k^o + q^o|) \mathcal{P} \left[\frac{1}{q^2} \right] \right\} T^{\mu\nu,\alpha\beta}(\vec{q}, \vec{k})
\end{aligned}$$

where μ is a simple redefinition of the renormalization parameter $\bar{\mu}$ given by $\mu \equiv \bar{\mu} \exp(\frac{23}{15} + \frac{1}{2} \ln 4\pi - \frac{1}{2}\gamma)$, and the tensors $Q^{\mu\nu,\alpha\beta}(k)$ and $\bar{Q}^{\mu\nu,\alpha\beta}(k)$ are defined, respectively, by

$$\begin{aligned}
Q^{\mu\nu,\alpha\beta}(k) &= \frac{3}{2} \left\{ T_1^{\mu\nu,\alpha\beta}(q, k) - \frac{1}{k^2} T_8^{\mu\nu,\alpha\beta}(q, k) + \frac{2}{k^4} T_{12}^{\mu\nu,\alpha\beta}(q, k) \right\} \\
&\quad - [1 - 360(\xi - \frac{1}{6})^2] \left\{ T_4^{\mu\nu,\alpha\beta}(q, k) + \frac{1}{k^4} T_{12}^{\mu\nu,\alpha\beta}(q, k) - \frac{1}{k^2} T_{13}^{\mu\nu,\alpha\beta}(q, k) \right\}, \quad (5.26)
\end{aligned}$$

$$\bar{Q}^{\mu\nu,\alpha\beta}(k) = [1 + 576(\xi - \frac{1}{6})^2 - 60(\xi - \frac{1}{6})(1 - 36\xi')] \left\{ T_4^{\mu\nu,\alpha\beta}(q, k) + \frac{1}{k^4} T_{12}^{\mu\nu,\alpha\beta}(q, k) - \frac{1}{k^2} T_{13}^{\mu\nu,\alpha\beta}(q, k) \right\}. \quad (5.27)$$

In the above and subsequent equations, we denote the coupling parameter in four dimensions $\xi(4)$ by ξ and consequently ξ' means $d\xi(n)/dn$ evaluated at $n = 4$. $\tilde{H}^{\mu\nu,\alpha\beta}(k)$ is the complete contribution of a free massless quantum scalar field to the thermal graviton polarization tensor [125, 126] and it is responsible for the instabilities found in flat spacetime at finite temperature [124, 125, 126]¹⁰. Eq. (5.25) reflects the fact that the kernel $\tilde{H}^{\mu\nu,\alpha\beta}(k)$ has thermal as well as non-thermal contributions. Note that it reduces to the first term in the zero temperature limit ($\beta \rightarrow \infty$)

$$\tilde{H}^{\mu\nu,\alpha\beta}(k) \simeq -\frac{\alpha k^4}{4} \left\{ \frac{1}{2} \ln \frac{|k^2|}{\mu^2} Q^{\mu\nu,\alpha\beta}(k) + \frac{1}{3} \bar{Q}^{\mu\nu,\alpha\beta}(k) \right\}. \quad (5.28)$$

and at high temperatures the leading term (β^{-4}) may be written as

$$\tilde{H}^{\mu\nu,\alpha\beta}(k) \simeq \frac{\pi^2}{30\beta^4} \sum_{i=1}^{14} H_i(r) T_i^{\mu\nu,\alpha\beta}(u, K), \quad (5.29)$$

where we have introduced the dimensionless external momentum $K^\mu \equiv k^\mu/|\vec{k}| \equiv (r, \hat{k})$. The $H_i(r)$ coefficients were first given in [125] and generalized to the next-to-leading order (β^{-2}) in [126]. (They are given with the MTW sign convention in [72].)

Finally, as defined above, $N^{\mu\nu,\alpha\beta}(x)$ is the noise kernel representing the random fluctuations of the thermal radiance and $D^{\mu\nu,\alpha\beta}(x)$ is the dissipation kernel, describing the dissipation of energy of the gravitational field.

5.4 Near Horizon Case

In this case, since the perturbation is taken around the Schwarzschild spacetime, exact expressions for the corresponding unperturbed propagators $G_{ab}^\beta[h_{\mu\nu}^\pm]$ are not known. Therefore apart from the

¹⁰Note that the addition of the contribution of other kinds of matter fields to the effective action, even graviton contributions, does not change the tensor structure of these kernels and only the overall factors are different to leading order [125].

approximation of computing the CTP effective action to certain order in perturbation theory, an appropriate approximation scheme for the unperturbed Green's functions is also required. This feature manifested itself in York's calculation of backreaction as well, where, in writing the $\langle T_{\mu\nu} \rangle$ on the right hand side of the semiclassical Einstein equation in the unperturbed Schwarzschild metric, he had to use an approximate expression for $\langle T_{\mu\nu} \rangle$ in the Schwarzschild metric given by Page [103]. The additional complication here is that while to obtain $\langle T_{\mu\nu} \rangle$ as in York's calculation, the knowledge of only the thermal Feynman Green's function is required, to calculate the CTP effective action one needs the knowledge of the full matrix propagator, which involves the Feynman, Schwinger and Wightman functions.

It is indeed possible to construct the full thermal matrix propagator $G_{ab}^{\beta}[h_{\mu\nu}^{\pm}]$ based on Page's approximate Feynman Green's function by using identities relating the Feynman Green's function with the other Green's functions with different boundary conditions. One can then proceed to explicitly compute a CTP effective action and hence the influence functional based on this approximation. However, we desist from delving into such a calculation for the following reason. Our main interest in performing such a calculation is to identify and analyze the noise term which is the new ingredient in the backreaction. We have mentioned that the noise term gives a stochastic contribution $\xi^{\mu\nu}$ to the Einstein-Langevin equation (2.12). We had also stated that this term is related to the variance of fluctuations in $T_{\mu\nu}$, i.e, schematically, to $\langle T_{\mu\nu}^2 \rangle$. However, a calculation of $\langle T_{\mu\nu}^2 \rangle$ in the Hartle-Hawking state in a Schwarzschild background using the Page approximation was performed by Phillips and Hu [24, 25, 26] and it was shown that though the approximation is excellent as far as $\langle T_{\mu\nu} \rangle$ is concerned, it gives unacceptably large errors for $\langle T_{\mu\nu}^2 \rangle$ at the horizon. In fact, similar errors will be propagated in the non-local dissipation term as well, because both terms originate from the same source, that is, they come from the last trace term in (5.17) which contains terms quadratic in the Green's function. However, the Influence Functional or CTP formalism itself does not depend on the nature of the approximation, so we will attempt to exhibit the general structure of the calculation without resorting to a specific form for the Greens function and conjecture on what is to be expected. A more accurate computation can be performed using this formal structure once a better approximation becomes available.

The general structure of the CTP effective action arising from the calculation of the traces in equation (5.17) remains the same. But to write down explicit expressions for the non-local kernels one requires the input of the explicit form of $G_{ab}^{\beta}[h_{\mu\nu}^{\pm}]$ in the Schwarzschild metric, which is not available in closed form. We can make some general observations about the terms in there. The first line containing L does not have an explicit Fourier representation as given in the far field case, neither will $T_{(\beta)}^{\mu\nu}$ in the second line representing the zeroth order contribution to $\langle T_{\mu\nu} \rangle$ have a perfect fluid form. The third and fourth terms containing the remaining quadratic component of the real part of the effective action will not have any simple or even complicated analytic form. The symmetry properties of the kernels $H^{\mu\nu,\alpha\beta}(x, x')$ and $D^{\mu\nu,\alpha\beta}(x, x')$ remain intact, i.e., they are respectively even and odd in x, x' . The last term in the CTP effective action gives the imaginary part of the effective action and the kernel $N(x, x')$ is symmetric.

Continuing our general observations from this CTP effective action, using the connection between this thermal CTP effective action to the influence functional [69, 38] via an equation in the schematic form (3.3). We see that the nonlocal imaginary term containing the kernel $N^{\mu\nu,\alpha\beta}(x, x')$ is responsible for the generation of the stochastic noise term in the Einstein-Langevin equation and the real non-local term containing kernel $D^{\mu\nu,\alpha\beta}(x, x')$ is responsible for the non-local dissipation term. To derive the Einstein-Langevin equation we first construct the stochastic effective

action (3.13). We then derive the equation of motion, as shown earlier in (3.15), by taking its functional derivative with respect to $[h_{\mu\nu}]$ and equating it to zero. With the identification of noise and dissipation kernels, one can write down a linear, non-local relation of the form,

$$N(t - t') = \int d(s - s') K(t - t', s - s') \gamma(s - s'), \quad (5.30)$$

where $D(t, t') = -\partial_t \gamma(t, t')$. This is the general functional form of a Fluctuation-Dissipation relation (FDR) and $K(t, s)$ is called the fluctuation-dissipation kernel [87]. In the present context this relation depicts the backreaction of thermal Hawking radiance for a black hole in quasi-equilibrium.

5.5 Einstein-Langevin equation

In this section we show how a semiclassical Einstein-Langevin equation can be derived from the previous thermal CTP effective action. This equation depicts the stochastic evolution of the perturbations of the black hole under the influence of the fluctuations of the thermal scalar field.

The influence functional $\mathcal{F}_{\text{IF}} \equiv \exp(iS_{\text{IF}})$ previously introduced in Eq. (3.2) can be written in terms of the the CTP effective action $S_{\text{eff}}^\beta[h_{\mu\nu}^\pm]$ derived in equation (5.22) using Eq.(3.3). The Einstein-Langevin equation follows from taking the functional derivative of the stochastic effective action (3.13) with respect to $[h_{\mu\nu}](x)$ and imposing $[h_{\mu\nu}](x) = 0$. This leads to

$$\frac{1}{\ell_P^2} \int d^4 x' L_{(o)}^{\mu\nu, \alpha\beta}(x - x') h_{\alpha\beta}(x') + \frac{1}{2} T_{(\beta)}^{\mu\nu} + \int d^4 x' \left(H^{\mu\nu, \alpha\beta}(x - x') - D^{\mu\nu, \alpha\beta}(x - x') \right) h_{\alpha\beta}(x') + \xi^{\mu\nu}(x) = 0. \quad (5.31)$$

where

$$\langle \xi^{\mu\nu}(x) \xi^{\alpha\beta}(x') \rangle_j = N^{\mu\nu, \alpha\beta}(x - x'), \quad (5.32)$$

In the far field limit this equation should reduce to that obtained by Campos and Hu [72]: For gravitational perturbations $h^{\mu\nu}$ defined in (5.13) under the harmonic gauge $\bar{h}^{\mu\nu}_{,\nu} = 0$, their Einstein-Langevin equation is given by

$$\square \bar{h}^{\mu\nu}(x) + \frac{1}{(16\pi G_N^2)} \left\{ T_{(\beta)}^{\mu\nu} + 2P_{\rho\sigma, \alpha\beta} \int d^4 x' \left(H^{\mu\nu, \alpha\beta}(x - x') - D^{\mu\nu, \alpha\beta}(x - x') \right) \bar{h}^{\rho\sigma}(x') + 2\xi^{\mu\nu}(x) \right\} = 0, \quad (5.33)$$

where the tensor $P_{\rho\sigma, \alpha\beta}$ is given by

$$P_{\rho\sigma, \alpha\beta} = \frac{1}{2} (\eta_{\rho\alpha} \eta_{\sigma\beta} + \eta_{\rho\beta} \eta_{\sigma\alpha} - \eta_{\rho\sigma} \eta_{\alpha\beta}). \quad (5.34)$$

The expression for $P_{\rho\sigma, \alpha\beta}$ in the near horizon limit of course cannot be expressed in such a simple form. Note that this differential stochastic equation includes a non-local term responsible for the dissipation of the gravitational field and a noise source term which accounts for the fluctuations of the quantum field. Note also that this equation in combination with the correlation for the stochastic variable (5.32) determine the two-point correlation for the stochastic metric fluctuations $\langle \bar{h}_{\mu\nu}(x) \bar{h}_{\alpha\beta}(x') \rangle_\xi$ self-consistently.

As we have seen before and here, the Einstein-Langevin equation is a dynamical equation governing the dissipative evolution of the gravitational field under the influence of the fluctuations of the quantum field, which, in the case of black holes, takes the form of thermal radiance. From

its form we can see that even for the quasi-static case under study the back reaction of Hawking radiation on the black hole spacetime has an innate dynamical nature.

For the far field case making use of the explicit forms available for the noise and dissipation kernels Campos and Hu [72] formally proved the existence of a Fluctuation-Dissipation Relation (FDR) at all temperatures between the quantum fluctuations of the thermal radiance and the dissipation of the gravitational field. They also showed the formal equivalence of this method with Linear Response Theory (LRT) for lowest order perturbation of a near-equilibrium system, and how the response functions such as the contribution of the quantum scalar field to the thermal graviton polarization tensor can be derived. An important quantity not usually obtained in LRT, but of equal importance, manifest in the CTP stochastic approach is the noise term arising from the quantum and statistical fluctuations in the thermal field. The example given in this section shows that the back reaction is intrinsically a dynamic process described (at this level of sophistication) by the Einstein-Langevin equation. By comparison, traditional LRT calculations cannot capture the dynamics as fully and thus cannot provide a complete description of the backreaction problem.

5.6 Discussions

We now draw some connection with related work. As remarked earlier, except for the near-flat case, an analytic form of the Green function is not available. Even the Page approximation [103] which gives unexpectedly good results for the stress energy tensor has been shown to fail in the fluctuations of the energy density [25, 26]. Thus using such an approximation for the noise kernel will give unreliable results for the Einstein-Langevin equation. If we confine ourselves to Page's approximation and derive the equation of motion without the stochastic term, we expect to recover York's semiclassical Einstein's equation if one retains only the zeroth order contribution, i.e, the first two terms in the expression for the CTP effective action in Eq. (5.22). Thus, this offers a new route to arrive at York's semiclassical Einstein's equations. Not only is it a derivation of York's result from a different point of view, but it also shows how his result arises as an appropriate limit of a more complete framework, i.e, it arises when one averages over the noise. Another point worth noting is that our treatment will also yield a non-local dissipation term arising from the fourth term in equation (5.22) in the CTP effective action which is absent in York's treatment. This difference arises primarily due to the difference in the way backreaction is treated, at the level of iterative approximations on the equation of motion as in York, versus the treatment at the effective action level as pursued here. In York's treatment, the Einstein tensor is computed to first order in perturbation theory, while $\langle T_{\mu\nu} \rangle$ on the right hand side of the semiclassical Einstein equation is replaced by the zeroth order term. In the effective action treatment the full effective action is computed to second order in perturbation, and hence includes the higher order non-local terms.

The other important conceptual point that comes to light from this approach is that related to the Fluctuation-Dissipation Relation. In the quantum Brownian motion analog (e.g., [87] and references therein), the dissipation of the energy of the Brownian particle as it approaches equilibrium and the fluctuations at equilibrium are connected by the Fluctuation - Dissipation relation. Here the backreaction of quantum fields on black holes also consists of two forms – dissipation and fluctuation or noise, corresponding to the real and imaginary parts of the influence functional as embodied in the dissipation and noise kernels. A FDR relation has been shown to exist for the near flat case by Campos and Hu [72] and we anticipate that it should also exist between the noise and dissipation kernels for the general case, as it is a categorical relation [87, 114]. Martin and

Verdaguer have also proved the existence of a FDR when the semiclassical background is a stationary spacetime and the quantum field is in thermal equilibrium. Their result was then extended to a conformal field in a conformally stationary background [27]. The existence of a FDR for the black hole case has been discussed by some authors previously [118, 119]. In [115], Hu, Raval and Sinha have described how this approach and results differ from those of previous authors. The FDR reveals an interesting connection between black holes interacting with quantum fields and non-equilibrium statistical mechanics. Even in its restricted quasi-static form, this relation will allow us to study *nonequilibrium* thermodynamic properties of the black hole under the influence of stochastic fluctuations of the energy momentum tensor dictated by the noise terms.

There are limitations of a technical nature in the specific example invoked here. For one we have to confine ourselves to small perturbations about a background metric. For another, as mentioned above, there is no reliable approximation to the Schwarzschild thermal Green's function to explicitly compute the noise and dissipation kernels. This limits our ability to present explicit analytical expressions for these kernels. One can try to improve on Page's approximation by retaining terms to higher order. A less ambitious first step could be to confine attention to the horizon and using approximations that are restricted to near the horizon and work out the Influence Functional in this regime.

Yet another technical limitation of the specific example is the following. Though we have allowed for backreaction effects to modify the initial state in the sense that the temperature of the Hartle-Hawking state gets affected by the backreaction, we have essentially confined our analysis to a Hartle-Hawking thermal state of the field. This analysis does not directly extend to a more general class of states, for example to the case where the initial state of the field is in the Unruh vacuum. Thus we will not be able to comment on issues of the stability of an *isolated* radiating black hole under the influence of stochastic fluctuations.

6 Further Developments

In this review we have given two routes to the establishment of stochastic gravity theory with derivation of the influence functional and the Einstein-Langevin equation (ELE). We also showed two examples of how backreaction problems can be treated by the influence functional or the CTP effective action method leading to an ELE describing the dissipative dynamics of spacetime driven by the fluctuations of the quantum field. In both of these examples we considered linear perturbations off a background spacetime, where the Green function for the quantum scalar field is readily available and the calculation of the CTP effective action can be performed by a perturbative expansion. This is true for the family of cosmological spacetimes, because of its high symmetry, and for the far field limit of the Schwarzschild spacetime. In the near horizon case, calculation is handicapped because an analytic form of the Green function is not available. These limitations are of a technical nature. However, the formulation of the problem and the overall strategy of attack are general enough to be applicable to a wide range of fluctuations and backreaction problems. This is the context where stochastic gravity theory was historically invented and where it can best be utilized.

We mention a number of ongoing research related to the topics discussed in this review. On the theory side, Roura and Verdaguer [50] has recently showed how stochastic gravity can be related to the large N limit of quantum metric fluctuations. Given N free matter fields weakly interacting with the gravitational field, Hartle and Horowitz [56] and Tomboulis [57] have shown

that semiclassical gravity can be obtained as the leading order large N limit (while keeping N times the gravitational coupling constant fixed). It is of interest to find out where in this setting can one place the fluctuations of the quantum fields and the metric fluctuations they induce; specifically, whether the inclusion of these sources will lead to an Einstein-Langevin equation [38, 39, 40, 41, 42], as it was derived historically in other ways as described in the first part of this review. This is useful because it may provide another pathway or angle in connecting semiclassical to quantum gravity (Using interacting quantum fields as example, a related idea is the kinetic approach to quantum gravity described in [60]).

6.1 Metric Fluctuations and Structure Formation

Starobinsky’s stochastic inflation [130] is an interesting paradigm for treating structure formation in that the classical long wavelength modes governed by a stochastic equation is driven by a noise originating from the quantum fluctuations of the high frequency modes. However, there are conceptual and technical problems (see, e.g., [131, 38, 132]). Specifically, how the long wavelength modes turn classical and how the quantum fluctuations act like noise. These bear on the issues of decoherence of the mean field and the quantum field origin of noise, two issues at the foundations of all theories of structure formation based on quantum fluctuations. Stochastic gravity is closest in spirit to this paradigm and is thus expected to be the proper theoretical framework for addressing these outstanding issues.

In another related problem, standard theories of structure formation from quantum (inflaton) fields and their fluctuations are based on the quantization of the linear perturbations of both the metric and the inflaton field [129]. A question naturally arises for stochastic gravity as to whether one can recover the same equations for the quantum correlation functions with the correlation function of the metric fluctuations. Recently Roura and Verdaguer [30] have given an expression of the two point function of metric perturbations in terms of the stochastic source from the products of the Hadamard functions of the inflaton field in a quasi- de Sitter spacetime, which is connected to the stress energy tensor fluctuations through the noise kernel. Although the gravitational fluctuations are assumed to be classical in stochastic gravity, at least in the linear regime, their correlation functions predicted by the Einstein-Langevin equation gives the correct symmetrized quantum two point functions [53]. This actually simplifies the conventional approach. Another advantage of the stochastic gravity approach is that it can also tackle gravitational fluctuations in inflationary models which are not driven by an inflaton field, but by vacuum polarization effects such as the trace anomaly (e.g., in Starobinsky inflation [88, 89, 90]).

6.2 Metric Fluctuations in Black Holes

In addition to the work described above by Campos, Hu, Raval and Sinha [72, 115, 116] and earlier work quoted therein, we mention also some recent work on black hole metric fluctuations and their effect on Hawking radiation. For example, Casher et al [133] and Sorkin [134] have concentrated on the issue of fluctuations of the horizon induced by a fluctuating metric. Casher et al [133] considers the fluctuations of the horizon induced by the “atmosphere” of high angular momentum particles near the horizon, while Sorkin [134] calculates fluctuations of the shape of the horizon induced by the quantum field fluctuations under a Newtonian approximations. Both group of authors come to the conclusion that horizon fluctuations become large at scales much larger than the Planck scale (note Ford and Svaiter [136] later presented results contrary to this

claim). However, though these works do deal with backreaction, the fluctuations considered do not arise as an explicit stochastic noise term as in our treatment. It may be worthwhile exploring the horizon fluctuations induced by the stochastic metric in our model and comparing the conclusions with the above authors. Barrabes et al [135] have considered the propagation of null rays and massless fields in a black hole fluctuating geometry and have shown that the stochastic nature of the metric leads to a modified dispersion relation and helps to confront the trans-Planckian frequency problem. However, in this case the stochastic noise is put in by hand and does not naturally arise from coarse graining as in the quantum open systems approach. It also does not take backreaction into account. It will be interesting to explore how a stochastic black hole metric, arising as a solution to the Einstein-Langevin equation, hence fully incorporating backreaction, would affect the trans-Planckian problem.

Ford and his collaborators [136, 34, 137] have also explored the issue of metric fluctuations in detail and in particular have studied the fluctuations of the black hole horizon induced by metric fluctuations. However, the fluctuations they considered are in the context of a fixed background and do not relate to the backreaction.

Another work originating from the same vein of stochastic gravity but not complying with the backreaction spirit is that of Hu and Shiokawa [138], who study novel effects associated with electromagnetic wave propagation in a Robertson-Walker universe and the Schwarzschild space-time with a small amount of given metric stochasticity. For the Schwarzschild metric, they find that time-independent randomness can decrease the total luminosity of Hawking radiation due to multiple scattering of waves outside the black hole and gives rise to event horizon fluctuations and fluctuations in the Hawking temperature. The stochasticity in a background metric in their work is assumed rather than derived (from quantum field fluctuations, as in this work) and so is not in the same spirit of backreaction. But it is interesting to compare their results with that of backreaction, so one can begin to get a sense of the different sources of stochasticity and their weights (see, e.g., [1] for a list of possible sources of stochasticity.)

In a subsequent paper Shiokawa [139] showed that the scalar and spinor waves in a stochastic spacetime behave similarly to the electrons in a disordered system. Viewing this as a quantum transport problem, he expressed the conductance and its fluctuations in terms of a nonlinear sigma model in the closed time path formalism and showed that the conductance fluctuations are universal, independent of the volume of the stochastic region and the amount of stochasticity. This result can have significant importance in characterizing the mesoscopic behavior of spacetimes resting between the semiclassical and the quantum regimes.

The stochastic approach to the study of black hole backreaction thus has a very rich structure and opens up many new avenues of inquiry. In particular it provides the proper platform and framework to launch a new program of research into the *nonequilibrium black hole thermodynamics*.

As illustrated in the cosmological backreaction example stochastic gravity theory can be applied to quasi-dynamic or even fully dynamic problems such as black hole collapse, technical difficulty of finding reasonable analytic approximations of the Green function or numerical evaluation of mode-sums notwithstanding. Our discussion shows that stochastic gravity based on open systems concepts and the close-time-path or influence functional methods is the preferred framework for backreaction problems of dynamical spacetimes interacting with quantum fields and is specially suitable for treating spacetime fluctuations and non-equilibrium conditions of matter fields.

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